The heart of this proof is doing something to "cancel out" the unwanted z's. Adding $-z$ to both members in step b does this job. The remainder of the proof consists of tidying up the resulting equation. This property is sometimes called the *Cancellation Property of Equality for Addition.*

A theorem used to make the proof of a subsequent theorem easier is called a *lemma* for that theorem. A theorem which follows directly from a previous theorem is called a *corollary* of that theorem. Thus, the addition property of equality is used as a lemma for proving its converse.

If both the theorem and its converse are true, you may write both as a *single* statement. You use the words "if and only if":

$$x + z = y + z \textit{ if and only if } x = y.$$

EXAMPLE 4

Adding Like Terms

Prove, for example, that $2x + 3x = 5x$.

Proof:

a.	$2x + 3x$	
	$= (2 + 3)x$	Distributivity
b.	$= 5x$	Arithmetic
c.	$\therefore 2x + 3x = 5x$	Transitivity

In this proof you started with *one member* of the equation in the conclusion, and transformed it to the *other* member. The transitive step at the end connects the original expression with the final one, thus expressing the desired conclusion.

These four examples illustrate three slightly different techniques for proving theorems.

THEOREM-PROVING TECHNIQUES

1. Start with one member of the desired equation and transform it to the other member (Example 4).
2. Start with a given equation and transform it to the desired equation (Example 3).
3. Start somewhere else and use a clever series of transformations or a clever argument (Examples 1 and 2).

The following exercise contains most of the basic properties you learned in previous mathematics courses. Your main purpose in proving these properties is so that you will realize that they are all based on the *axioms*. You must also, of course, remember *what* they say so that you will be able to use them later on.

EXERCISE 1-7

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in les than 5 minutes.

Q1. Solve: $(x - 3) = 9$

Q2. Simplify: $(x - 3) - 9$

Q3. Commute the two factors: $(x + 4)(x - 7)$

Q4. Distribute: $4x(x + 13)$

Q5. Write the additive identity element.

Q6. Find 7% of 700.

Q7. Evaluate $|3x - 29|$ if x is 4.

Q8. Is -3000 an even number?

Q9. Is $\sqrt{-9}$ a real number?

Q10. Add: $4.7 + 3$

Work the following problems.

1. Write an example of each of the following axioms:
   a. Transitivity for equality.
   b. Transitivity for order.
   c. Symmetry for equality.
   d. Reflexive axiom for equality.
   e. Trichotomy.
2. Explain why the order relationship "$<$" is *not* symmetric and *not* reflexive.
3. What is the difference between an axiom and any other property?
4. Calvin Butterball sees the expression $x^2 + 3x - 5$. He decides to substitute 7 for the first x and 2 for the second x. What axiom has he violated?

5. What axiom tells you that a variable such as x can stand for only *one* number at a time?

6. State, without proof,
   a. the Addition Property of Order, and
   b. the Multiplication Property of Order.

For Problems 7 through 32, prove the property either from scratch or by supplying names of the properties justifying the given steps. As reasons for steps, you may use any axiom or definition, or any other property whose proof appears *before* the one you are doing.

7. Prove the *Multiplication Property of Equality* which states that you can multiply both members of an equation by the same number.

8. Prove the *converse* of the Multiplication Property of Equality.

9. If you proved the property in Problem 8, you proved a *false* theorem! For example, $3 \times 0 = 5 \times 0$, but 3 does *not* equal 5. Fix the hypothesis in Problem 8 so that the theorem *is* true. Then think up a good name for this property.

10. Can the Multiplication Property of Equality and its converse be written as a single statement using "if and only if"? Explain.

11. The symbol $-(-x)$ means the additive inverse of $-x$. Use this fact to prove that $-(-x) = x$.

12. The symbol $\frac{1}{\frac{1}{x}}$ means the multiplicative inverse (reciprocal) of $\frac{1}{x}$. Use this fact to prove that $\frac{1}{\frac{1}{x}} = x$. For what value of x is the theorem *false*?

13. *Property of the Reciprocal of a Product*

    Prove that $\frac{1}{xy} = \frac{1}{x} \cdot \frac{1}{y}$.

    Proof:

    a. $\left(\frac{1}{xy}\right)(xy) = 1$

    b. $\left[\left(\frac{1}{xy}\right)(xy)\right] \cdot \frac{1}{y} = 1 \cdot \frac{1}{y}$

    c. $\left[\left(\frac{1}{xy}\right)(x)\right]\left(y \cdot \frac{1}{y}\right) = 1 \cdot \frac{1}{y}$

d. $\left[\left(\frac{1}{xy}\right)(x)\right] \cdot 1 = 1 \cdot \frac{1}{y}$

e. $\left(\frac{1}{xy}\right)(x) = \frac{1}{y}$

f. $\left[\left(\frac{1}{xy}\right)(x)\right] \cdot \frac{1}{x} = \frac{1}{y} \cdot \frac{1}{x}$

g. $\frac{1}{xy}\left(x \cdot \frac{1}{x}\right) = \frac{1}{y} \cdot \frac{1}{x}$

h. $\frac{1}{xy} \cdot 1 = \frac{1}{y} \cdot \frac{1}{x}$

i. $\frac{1}{xy} = \frac{1}{y} \cdot \frac{1}{x}$

j. $\therefore \frac{1}{xy} = \frac{1}{x} \cdot \frac{1}{y}$, Q.E.D.

14. ***Multiplication Property of Zero***

Prove that for any real number x, $x \cdot 0 = 0$.

Proof:

a. $0 = 0$
b. $0 + 0 = 0$
c. $x(0 + 0) = x \cdot 0$
d. $x(0 + 0) = 0 + x \cdot 0$
e. $x \cdot 0 + x \cdot 0 = 0 + x \cdot 0$
f. $\therefore x \cdot 0 = 0$, Q.E.D.

15. ***Converse of the Multiplication Property of Zero***

Prove that if $xy = 0$, then $x = 0$ or $y = 0$.

Proof:

a. Either $y = 0$ or $y \neq 0$

b. If $y = 0$, the conclusion is true.
(Only *one* clause of an "or" statement needs to be true.)

c. If $y \neq 0$, then $\frac{1}{y}$ is a real number.

d. $xy = 0$

e. $(xy) \cdot \frac{1}{y} = 0 \cdot \frac{1}{y}$

f. $(xy) \cdot \frac{1}{y} = 0$

g. $x\left(y \cdot \frac{1}{y}\right) = 0$

h. $x \cdot 1 = 0$

i. $x = 0$

j. $\therefore x = 0$ or $y = 0$, Q.E.D.
(From Steps b and i.)

16. The Multiplication Property of Zero can be stated, "If $x = 0$ or $y = 0$, then $xy = 0$." Write this property and its converse as a *single* statement using "if and only if" terminology.

17. *Lemma—Multiplication Property of Negative One*

Prove that $-1 \cdot x = -x$.

Proof:

$-1 \cdot x + x$
a. $= -1 \cdot x + 1 \cdot x$
b. $= (-1 + 1)(x)$
c. $= 0 \cdot x$
d. $= 0$
e. $= -x + x$
f. $\therefore -1 \cdot x + x = -x + x$
g. $\therefore -1 \cdot x = -x$, Q.E.D.

18. *Theorem—The Product of Two Negatives is Positive.*

Prove that $(-x)(-y) = xy$.

Proof:

$(-x)(-y)$
a. $= (-1 \cdot x)(-1 \cdot y)$
b. $= (-1)[x \cdot (-1)](y)$
c. $= (-1)[-1 \cdot x](y)$
d. $= [-1 \cdot (-1)](xy)$
e. $= 1 \cdot xy$
f. $= xy$
g. $\therefore (-x)(-y) = xy$, Q.E.D.

Note that there *is* a reason why $-1 \cdot (-1) = 1$ in Step e which does not involve circular reasoning. You must treat each of the -1's *differently* to see how the reasoning works. This property tells you the *real* reason why the product of two negatives is positive. If it came out any other way, it would contradict the Field Axioms. This is an example of a not-so-obvious property which turns out to be *true*.

19. ***If Two Real Numbers are Equal, then their Additive Inverses are Equal.***
    a. *State* this property using letters to stand for the numbers.
    b. *Prove* this property. You may find that the Multiplication Property of -1 is useful as a lemma.

20. ***If Two Real Numbers are Equal, then their Multiplicative Inverses are Equal.***
    a. *State* this property using letters to stand for the numbers. Be sure that the hypothesis *excludes* the one number for which the property is false.
    b. *Prove* the property.

21. Prove that *the square of a real number is never negative*. Do this by showing that whatever value you pick for x,

$$x^2 \geq 0.$$

The axiom of trichotomy and the Multiplication Property of Order should be helpful as lemmas.

22. *Multiplication Distributes over Subtraction.*

Prove that $x(y - z) = xy - xz$.

Proof:

$x(y - z)$

a. $= x[y + (-z)]$
b. $= xy + x(-z)$
c. $= xy + x[(-1)(z)]$
d. $= xy + [(x)(-1)]z$
e. $= xy + [(-1)(x)]z$
f. $= xy + (-1)(xz)$
g. $= xy + (-xz)$
h. $= xy - xz$
i. $\therefore x(y - z) = xy - xz$, Q.E.D.

23. Prove that *division distributes over addition*. That is, prove that

$$\frac{x + y}{z} = \frac{x}{z} + \frac{y}{z}.$$

Do this by first transforming the division to multiplication using the Definition of Division. Then distribute the multiplication over the addition.

Note that this property when read backwards says that you can add two fractions when they have a *common* denominator. This explains why you must *find* a common denominator before you *can* add two fractions. Adding fractions any other way (such as adding the numerators and adding the denominators) would violate the field axioms.

24. *Multiplication Property of Fractions*

Prove that $\frac{xy}{ab} = \frac{x}{a} \cdot \frac{y}{b}$.

Proof:

a. $\frac{xy}{ab} = (xy) \cdot \frac{1}{ab}$

b. $= (xy)\left(\frac{1}{a} \cdot \frac{1}{b}\right)$

c. $= x\left(y \cdot \frac{1}{a}\right) \cdot \frac{1}{b}$

d. $= x\left(\frac{1}{a} \cdot y\right) \cdot \frac{1}{b}$

e. $= \left(x \cdot \frac{1}{a}\right)\left(y \cdot \frac{1}{b}\right)$

f. $= \frac{x}{a} \cdot \frac{y}{b}$

g. $\therefore \frac{xy}{ab} = \frac{x}{a} \cdot \frac{y}{b}$, Q.E.D.

Note that this property when read backwards says that the way to multiply two fractions is to multiply their denominators together and multiply their numerators together. Again, the reason you do not multiply fractions in other ways is because to do so would violate the field axioms.

25. Prove that *division distributes over subtraction.*

26. Prove that a *non-zero number divided by itself equals 1*, i.e., $\frac{n}{n} = 1$.

27. Prove that *1 is its own reciprocal.* That is, $\frac{1}{1} = 1$.

28. Prove that a *number divided by 1 is that number*, i.e., $\frac{n}{1} = n$.

29. Prove that a *negative number divided by a positive number is negative*, $\frac{-x}{y} = -\frac{x}{y}$.

30. Prove that a *positive number divided by a negative number is negative,* $\frac{x}{-y} = -\frac{x}{y}$.

31. Prove that *the negative of a sum equals the sum of the negatives.* That is, prove that $-(x + y) = -x + (-y)$.

32. Prove that $x - y$ *and* $y - x$ *are additive inverses of each other.* That is, prove that $-(x - y) = y - x$.

1-8 CHAPTER REVIEW AND TEST

The purpose of this chapter has been to refresh your memory about some of the words and techniques of mathematics so that you and your instructor will be speaking the same language. The objectives of the chapter can be summarized as follows:

1. Name various kinds of numbers.
2. State and prove properties.
3. a. Recognize and name polynomials.
b. Simplify expressions.
c. Evaluate expressions.
4. Solve equations and inequalities.

The Review Problems below give you a relatively straightforward test of these objectives. The problem numbers correspond to the objectives so that you will have no doubt about what is expected of you. The Concepts Test, on the other hand, has problems that may require you to use *several* of the objectives. In some cases, you will have the chance to *extend* your knowledge by applying what you know to *new* situations. For this reason, the Concepts Test may be longer and more difficult than a test your instructor might give you.

REVIEW PROBLEMS

The following problems are numbered according to the four objectives listed above.

R1. a. Give an example of
i. a rational number that is not an integer,
ii. an irrational number that is not positive,
iii. an imaginary number,
iv. a transcendental number,

v. a negative even number,
vi. a positive integer that is not a digit,
vii. a natural number,
viii. a real number that is not a natural number,
ix. a real number that is neither positive nor negative,
x. an irrational number that is not a real number.

b. Name all the sets of numbers to which each of the following belongs.
i. 2
ii. -3
iii. $\sqrt{3}$
iv. $\sqrt{-3}$
v. 2.3

R2. a. What is meant by
i. an axiom?
ii. a lemma?
iii. a corollary?
iv. a hypothesis?

b. State each of the following, and tell whether or not it is an axiom. If so, is it a *Field* Axiom?
i. Definition of Subtraction.
ii. Multiplicative Inverses Property.
iii. Trichotomy.
iv. Closure of {real numbers} under addition.
v. Additive Identity Property.
vi. Multiplication Property of Order.

c. *Positive Divided by Negative* — Justify each step of the following proof.

Prove that $\frac{x}{-y} = -\frac{x}{y}$.

Proof: $\frac{x}{-y}$

i. $= x \cdot \frac{1}{-y}$

ii. $= x \cdot \frac{1}{-1 \cdot y}$

iii. $= x \cdot \left(\frac{1}{-1} \cdot \frac{1}{y}\right)$

iv. $= x \cdot (-1 \cdot \frac{1}{y})$

v. $= [x \cdot (-1)] \cdot \frac{1}{y}$

vi. $= [-1 \cdot x] \cdot \frac{1}{y}$

vii. $= -1 \cdot \left(x \cdot \frac{1}{y}\right)$

viii. $= -1 \cdot \frac{x}{y}$

ix. $= -\frac{x}{y}$

x. $\therefore \frac{x}{-y} = -\frac{x}{y}$, Q.E.D.

d. The Distributive Axiom states that multiplication distributes over a sum of *two* terms. That is, $a(b + c) = ab + ac$. Prove that multiplication also distributes over a sum of *three* terms. Starting with $a(b + c + d)$, you can use the Associative Property to write the expression inside the parentheses as *two* terms. Then you can use the Distributive Axiom *twice* to get the desired result.

R3. a. Name each polynomial by degree and by number of terms. If it is *not* a polynomial, tell *why* not.

i. $x^2y - 5$
ii. $\frac{x^2}{y} - 5$
iii. $x^2\sqrt{y - 5}$
iv. $x^2y^2 - \sqrt{5}$
v. $4^2r^2s^3$
vi. $6x^2 + 7x - 5$

b. Carry out the indicated operations and simplify.

i. $13 - 5 + 1$
ii. $40 \div 10 \times 2$
iii. $24 - 12 \div 3 + 1$
iv. $(3x + 7)(x - 8)$
v. $5 - 3[x - 7(2x - 6)]$

c. Evaluate the following expressions for $x = 5$ and $x = -4$.

i. $3x - 8$
ii. $|2x - 10|$
iii. $3x^2 - 2x + 11$

R4. a. Write the solution set.

i. $2x + 7 = -5$, $x \in$ {integers}
ii. $5x - 3 = 8$, $x \in$ {integers}
iii. $(2x + 6)(3x - 2) = 0$, $x \in$ {rational numbers}
iv. $|4x + 3| = 9$, $x \in$ {positive numbers}
v. $x^2 = 81$, $x \in$ {real numbers}

b. Graph the solution set.
   i. $4x - 3 < 7$, $x \in$ {real numbers}
   ii. $2 - 5x \leq 17$, $x \in$ {real numbers}
   iii. $|x - 2| > 5$, $x \in$ {integers}
   iv. $|3 - 4x| \leq 9$, $x \in$ {integers}

CONCEPTS TEST

Work each of the problems below. For each part of each problem, tell by number and letter which one or ones of the above objectives you used in that problem. Also, if the problem involves a new concept, write, "new concept."

T1. Give an example of
  a. a cubic binomial with three variables,
  b. a quintic monomial with two variables,
  c. a quadratic trinomial,
  d. an expression that is not a polynomial,
  e. a rational number between -12 and -13,
  f. a radical that represents a rational number,
  g. the Multiplicative Identity Axiom,
  h. the Additive Inverse Axiom.

T2. Write an example that shows why,
  a. subtraction is *not* associative,
  b. exponentiation is *not* commutative,
  c. {real numbers} is *not* closed under the operation "square root,"
  d. multiplication *does* distribute over subtraction.

T3. Given the expression $3x - 4[2x - (5x - 9)]$,
  a. *evaluate* it by substituting 7 for x, and then doing the indicated operations,
  b. *simplify* it, *without* substituting a value for x,
  c. substitute 7 for x in the simplified expression of part b and show that you get the same value as in part a.

T4. Given the expression $(x + 7)(2x - 3)$,
  a. carry out the indicated multiplication and simplify,
  b. find the value(s) of x that make the expression equal to 0.

T5. Transform the following to equivalent inequalities that have *no* absolute value signs. Then graph the solution sets.
  a. $|x - 3| < 7$, $x \in$ {integers}
  b. $|7 - 2x| \leq 1$, $x \in$ {real numbers}

T6. What extraneous solution is created if you multiply both members of the equation $x = 17$ by the expression $(x - 23)$?

T7. There is a set of numbers that contains both the real numbers *and* the imaginary numbers. What is the name of this set?

T8. Write the Multiplication Property of Zero and its converse, as a *single* statement, using "if and only if."

T9. a. If the converse of the Multiplication Property of Equality were true, what would it say?
b. Explain why this converse is *false*.

T10. Professor Snarff gives an algebra test on which students must add

$$\frac{5}{7} + \frac{6}{7}.$$

Calvin Butterball gets $\frac{11}{14}$ and Phoebe Small gets $\frac{11}{7}$.
a. Who is right?
b. Name the property that explains *why* he or she is right.

T11. You have learned what it means to say that equality is "reflexive," "symmetric," and "transitive." Answer the following questions about the relationship " $\neq$," which means, "is *not* equal to."
a. Is $\neq$ reflexive? Justify your answer.
b. Is $\neq$ symmetric? Justify your answer.
c. Write an example that shows why $\neq$ is *not* transitive.

T12. In Section 1-7 it was stated that the Transitive Property of Order is an *axiom*. By making precise definitions of $>$ and $<$, you can use other axioms to *prove* that order is transitive.

DEFINITION

$>$ AND $<$

$a > b$ if and only if $a - b$ is *positive*.
$a < b$ if and only if $b > a$.

The only other fact you need to know is that {positive numbers} is *closed* under $+$ and $\times$. Supply reasons for the following proof.

Prove that if $x > y$ and $y > z$, then $x > z$.

Proof:

a. $x > y$ and $y > z$.
b. $x - y$ is positive and $y - z$ is positive.

c. $(x - y) + (y - z)$ is positive.
d. $x + (-y + y) - z$ is positive.
e. $x + 0 - z$ is positive.
f. $x - z$ is positive.
g. $x > z$, Q.E.D.

T13. Prove that if you multiply both members of an inequality by a *positive* number, the order does *not* change. That is, prove that if $x > y$ and z is positive, then $xz > yz$.

T14. Supply reasons that the order *reverses* in the steps in the following proof when you multiply both members of an inequality by a *negative* number.

Prove that if $x > y$ and z is negative, then $xz < yz$.

Proof:

a. z is negative.
b. $\therefore (-1)(z)$ is positive.
c. $\therefore -z$ is positive.
d. $x > y$
e. $\therefore x - y$ is positive.
f. $\therefore (x - y)(-z)$ is positive.
g. $\therefore -xz + yz$ is positive.
h. $\therefore yz + (-xz)$ is positive.
i. $\therefore yz - xz$ is positive.
j. $\therefore yz > xz$.
k. $\therefore xz < yz$, Q.E.D.

T15. Use the result of Problem Concepts Test 14 as a lemma to prove that if $x < y$ and z is negative, then $xz > yz$.