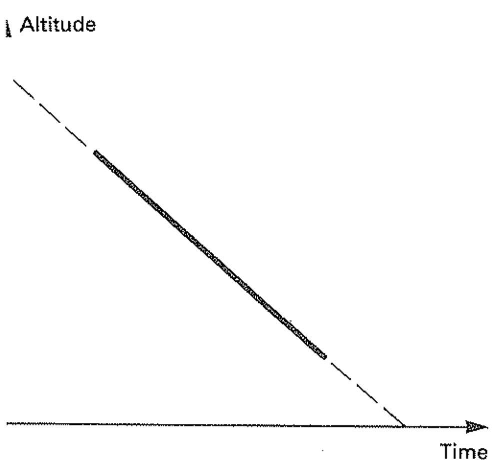
3

Linear Functions

*In Chapter 2 you learned that a **function** relates two variables. In this chapter you will study a special kind of function, the **linear** function. Your ultimate objective is to be able to write the **equation** for a linear function from given information about its graph. In Exercise 3-5 you will use this kind of equation to predict such things as how high a sky diver is.*



b. The length of time an oven has been turned on and the oven's temperature are related.
c. The angle at which you have to look up to see the Sun depends on the time of day. Make the domain extend for *several* days.
d. The number of cars in the student parking lot and the time of day are related.

CONCEPTS TEST

The following problems combine all of the objectives of this chapter. For each part of each problem, do what is asked, then identify by number which one or ones of the objectives you used in working that part.

T1. Consider the relation whose equation is

$$y = 1 + 4x - x^2.$$

a. Plot the graph of this relation, assuming that the domain is the set of non-negative numbers, and that the range must also contain only non-negative numbers.
b. What is the range of this relation?
c. Is the relation a function? Justify your answer.
d. At what value of y does the graph touch the y-axis?
e. At approximately what value of x does the graph touch the x-axis?
f. Tell *two* different real-world situations having graphs looking like this one.

T2. ***Introduction to Polynomial Functions*** A function is called a "polynomial function" if it has an equation of the form

$$y = \text{a polynomial involving } x.$$

a. Which of the relations in Problems 1–12 of Exercise 2-2 are polynomial functions?
b. Explain how closure insures that the domain of a polynomial function can be the set of *all* real numbers.
c. Explain how closure insures that a polynomial function really *is* a function.

3-1 INTRODUCTION TO LINEAR FUNCTIONS

A function is named according to its equation. For instance, if the equations is

$$y = 3x + 7,$$

then the function is called a *linear* function. This name is picked because y equals a linear polynomial in the variable x.

DEFINITION

LINEAR FUNCTION
A **linear function** is a function whose general equation is

$$y = mx + b,$$

where m and b stand for constants, and $m \neq 0$.

The equation $y = mx + b$ is called a *general equation*. If particular values are chosen for m and b, such as in $y = 3x + 7$, the equation is called a *particular equation*. If a particular equation had $m = 0$, such as $y = 7$, then y would equal a zero-degree polynomial. Thus, the function would be called a constant function, not a linear function.

Objective:
Discover what the graph of a linear function looks like, and what effects the values of m and b have.

In the following exercise you will accomplish this objective.

EXERCISE 3-1

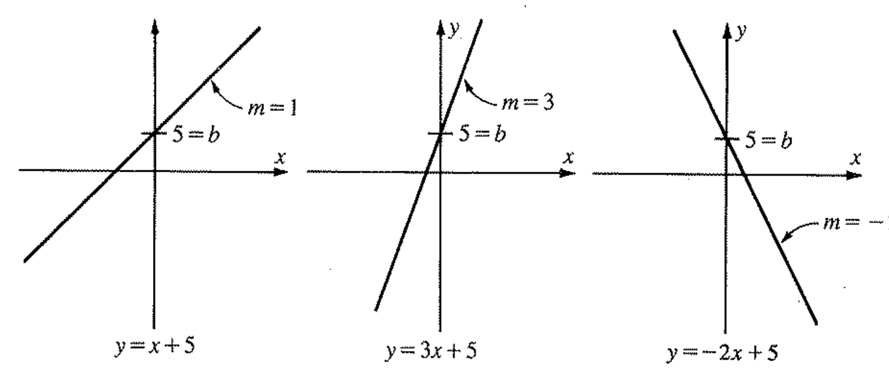
1. Plot the graphs of the following functions by selecting values of x, calculating the corresponding values of y, and plotting the points.
   a. $y = x + 5$ b. $y = 2x + 5$
   c. $y = 3x + 5$ d. $y = -2x + 5$
   e. $y = 2x - 5$ f. $y = 0x + 5$

2. From the graphs in Problem 1, answer the following questions.
   a. Why are first-degree functions called *linear* functions?
   b. What is the effect on the graph of changing the x-coefficient?
   c. What is the effect on the graph of changing the constant term?
   d. From what you have learned in previous mathematics courses, tell the special names given to the constants m and b in $y = mx + b$.

3. If m equals zero as in part (f) of Problem 1, the graph is a horizontal straight line. Yet the function is *not* called "linear." Why not?

3-2 PROPERTIES OF LINEAR FUNCTION GRAPHS

In Exercise 3-1 you plotted the graphs of some linear functions. Figure 3-2a shows what some of these graphs look like.



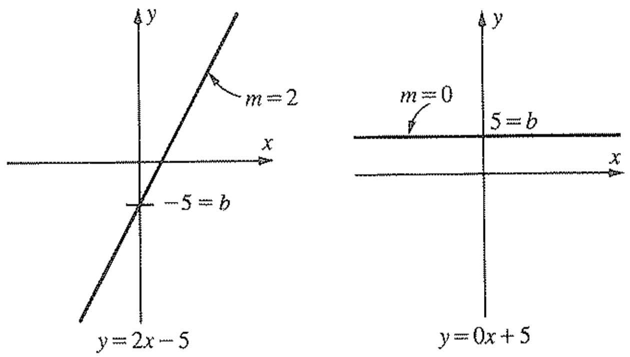


Figure 3-2a

From the graphs you should be able to see the following properties:

1. The graphs are *straight lines*.
2. The value of m determines how "tilted" the graph is:
   If m is positive: Graph slopes *up* as x increases.
   If m is negative: Graph slopes *down* as x increases.
   If m is zero: Graph is horizontal (constant function).
3. The value of b tells where the graph crosses the y-axis.

If you let x equal zero in an equation like $y = 3x + 7$, you get $y = 7$. So the value of y when $x = 0$ is the same as the constant term in the equation. Since this is the value of y where the graph "intercepts" the y-axis, it is called the y-intercept. Similarly, the x-intercept is the value of x when $y = 0$.

DEFINITION

INTERCEPTS
The **y-intercept** of a function is the value of y when $x = 0$.
An **x-intercept** of a function is a value of x when $y = 0$.

A property of linear function graphs is illustrated in Figure 3-2b. If you start at any point on the graph and run along in the positive direction, then rise up (or down) to another point on the graph, then the ratio

$$\frac{\text{rise}}{\text{run}}$$

will be *constant*, no matter what two points you pick. This property is a direct consequence of the properties of similar triangles that you learned in geometry.

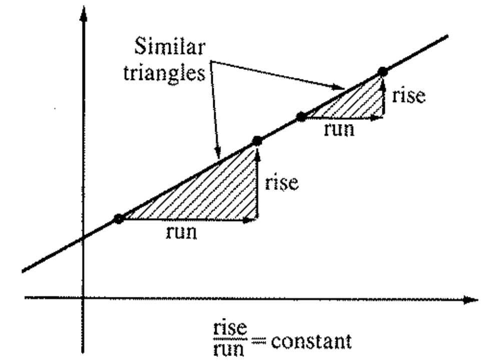


Figure 3-2b

The ratio rise/run is defined to be the *slope* of a linear function.

DEFINITION

SLOPE
The **slope** of a linear function is the ratio

$$\frac{\text{rise}}{\text{run}}$$

where the run is the horizontal distance between two points on the graph and the rise is vertical distance between the same two points.

If (x_1, y_1) and (x_2, y_2) are two points on the graph, then the rise and run can be found by subtracting the coordinates. Figure 3-2c shows why.

$$\text{rise} = y_2 - y_1 = \Delta y$$

$$\text{run} = x_2 - x_1 = \Delta x$$

The symbols Δy and Δx are pronounced "delta y" and "delta x." The Greek letter Δ is used because the rise and run are *differences* between y or x values.

Substituting these values into the definition of slope gives an equation that is called the *slope formula*.

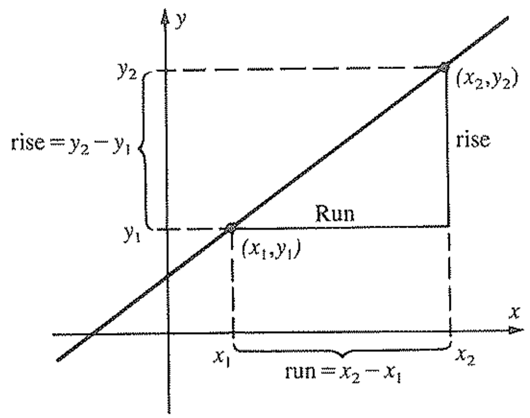


Figure 3-2c

PROPERTY

THE SLOPE FORMULA
If (x_1, y_1) and (x_2, y_2) are two points on the graph of a linear function, then

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

It turns out that the constant m in the equation is also equal to the slope. To see why, follow the steps in the proof below. You should supply reasons for the steps, as you did in Section 1-7.

THEOREM

If $y = mx + b$, then m is the slope.

Proof:
Let (x_1, y_1) and (x_2, y_2) be two points on the graph.
Then $y_2 = mx_2 + b$, and
$y_1 = mx_1 + b$.
Subtracting the bottom equation from the top one gives

$$y_2 - y_1 = mx_2 - mx_1$$

$$\therefore y_2 - y_1 = m(x_2 - x_1)$$

$$\therefore \frac{y_2 - y_1}{x_2 - x_1} = m$$

$\therefore m$ is the slope, Q.E.D.

PROPERTY

> **SLOPE-INTERCEPT FORM**
> If $y = mx + b$, then m equals the slope of the graph, and b equals the y-intercept.

Note that the intercepts and slope are *numbers* rather than geometrical features of the graph. It is more convenient to define these features as numbers so that you can do algebra with them.

Objective:

Given the particular equation of a linear function, plot its graph quickly, using slope and y-intercept.

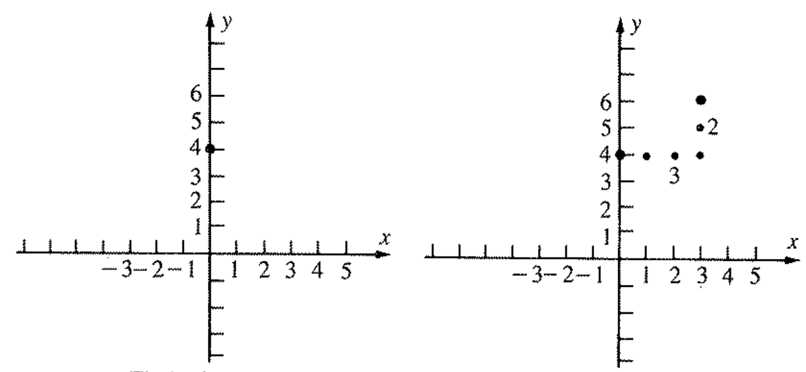
EXAMPLE 1

Plot the graph of $y = \frac{2}{3}x + 4$ quickly.

Solution:
Your thought process should be:

1. The y-intercept is 4, because $y = 4$ when $x = 0$. Put your pencil on the y-axis, 4 units up.
2. The slope is $\frac{2}{3}$. Run across 3 units to the right, then rise up 2 units. Mark another point on the graph. Repeat the process, if necessary, to get more points.
3. Connect the points with a straight line.

Figure 3-2d shows these three steps. ■



Find y-intercept

Find second point

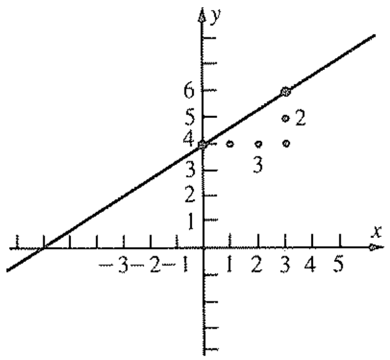


Figure 3-2d

EXAMPLE 2

Plot the graph of $5x + 7y = 14$ quickly.

Solution:
You can change this new problem into an "old" problem by transforming the equation to the form $y = mx + b$. You would write:

$$5x + 7y = 14$$
$$7y = -5x + 14$$
$$y = -\tfrac{5}{7}x + 2$$

So $m = -\frac{5}{7}$ and $b = 2$. Since the slope is a negative number, either the rise or the run must be negative (and the other must be positive). Starting at (0, 2) on the graph, you can either run forward 7 and down 5, or run backward 7 and up 5 to find another point. The graph is shown in Figure 3-2e.

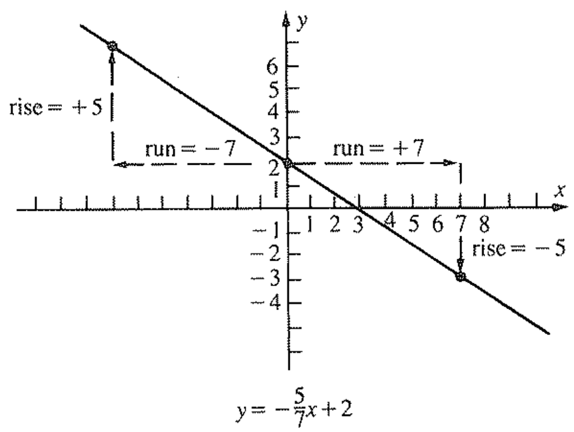


Figure 3-2e

EXAMPLE 3

Plot the graph of $x = 6$.

Solution:
This equation has the form $x + 0y = 6$. Transforming it to $y = mx + b$ form would give

$$y = -\frac{1}{0}x + \frac{6}{0}$$

The quantities $-\frac{1}{0}$ and $\frac{6}{0}$ are *infinite*, which means that they are larger than any real number. So there is no slope and no y-intercept. But drawing the graph is easy. Since x is 6 no matter what y is, the graph is a vertical line (Figure 3-2f). The relation is *not* a function, since there is more than one value of y when x is 6.

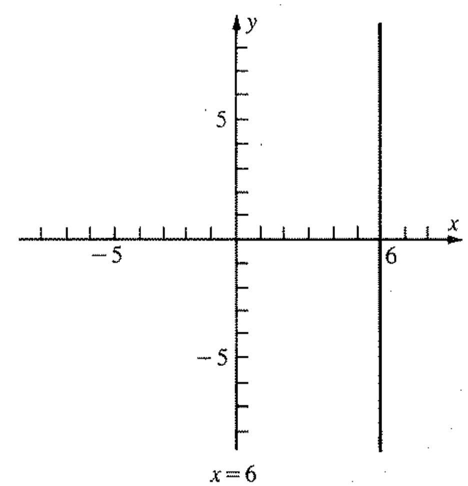


Figure 3-2f

If the equation in Example 3 had been $y = 7$, the graph would be a *horizontal* straight line. No matter what x is, y would always be 7. From this observation, a property can be concluded.

PROPERTY

HORIZONTAL AND VERTICAL LINES
If y = constant, then the graph is a horizontal straight line. The slope is 0.

If x = constant, then the graph is a vertical straight line. There is no number for the slope. The slope is infinitely large.

In the following exercise you will practice drawing linear graphs.

EXERCISE 3-2

Do These Quickly

The following problems are intended to refresh your skills from the first two chapters and from previous courses. You should be able to do all 10 in less than 5 minutes.

Q1. Write a rational number that is not an integer.

Q2. Write an integer that is not positive.

Q3. Write an odd prime number.

Q4. Solve: $3x + 7 = 31$

Q5. Evaluate $5x - 2$ if x is 3.

Q6. Find 20% of 63.

Q7. What axiom is illustrated: "If $x = y$, then $y = x$."

Q8. Evaluate $|2 - 5x|$ if x is 3.

Q9. Sketch the graph of a relation that is not a function.

Q10. Add $\frac{2}{3}$ and $\frac{3}{4}$.

Work these problems.

For Problems 1 through 20, plot the graph neatly on graph paper. Use the slope and y-intercept, where possible.

1. $y = \frac{3}{5}x + 3$
2. $y = \frac{5}{2}x - 1$
3. $y = -\frac{3}{2}x - 4$
4. $y = -\frac{1}{4}x + 3$
5. $y = 2x - 5$
6. $y = 3x - 2$
7. $y = -3x + 1$
8. $y = -2x + 6$
9. $7x + 2y = 10$
10. $3x + 5y = 10$
11. $x - 4y = 12$
12. $2x - 5y = 15$
13. $y = 3x$
14. $y = -2x$
15. $y = 3$
16. $y = -5$

17. $x = -4$

18. $x = 2$

19. $y = 0$

20. $x = 0$

21. Relations such as in Problems 15 through 20 where

$$x = \text{constant} \quad \text{or} \quad y = \text{constant}$$

are not called linear functions, even though their graphs are straight lines. However, the reason is different in each case. Explain why such relations are not called linear functions.

22. ***Intercepts Problem*** In the definition of intercepts, it says *the* y-intercept, but *an* x-intercept. Sketch a graph which shows that a function could have more than one x-intercept. Explain why a function could not have more than one y-intercept.

23. ***Division by Zero Problem*** Evaluate $\frac{1}{0.1}$, $\frac{1}{0.01}$, $\frac{1}{0.001}$, and $\frac{1}{0.0001}$. What happens to the size of a fraction as its denominator gets very close to zero? Why would $\frac{1}{0}$ be larger than any real number? What name is used for a quantity that is larger than any real number?

24. ***Slope Proof Problem*** Prove that if $y = mx + b$, where m and b are constants, then m is the slope. Write a reason to justify each step in the proof.

25. ***Another Form of the Linear Function Equation***
    a. Show that the relation

    $$y - 4 = 2(x - 5)$$

    is a *linear* function by transforming to $y = mx + b$.
    b. Plot the graph of the function.
    c. What does the number 2 in the original equation tell you about the graph?
    d. The coordinates of a point on the graph are concealed in the original equation! What point?

3-3 OTHER FORMS OF THE LINEAR FUNCTION EQUATION

Suppose that a relation has the equation

$$y - 4 = 2(x - 5).$$

If you substitute (5, 4) for (x, y), both members of the equation are zero. So (5, 4) is a point on the graph because it makes the equation a *true* statement. By distributing the 2, then adding 4 to each member, the equation becomes

$$y = 2x - 6.$$

So the relation is a *linear function* with slope 2. You may already have discovered this if you worked Problem 25 in Exercise 3-2.

An equation such as $y - 4 = 2(x - 5)$ is said to be in *point-slope* form because the coordinates of a point and the slope of the line appear in the equation. The familiar $y = mx + b$ form of the linear function equation is called *slope-intercept* form.

A linear function equation can be written as $3x + 4y = 13$. In this text, the name "$Ax + By = C$ form" is used if both variables are on one side, and the constant is on the other.

FORMS OF THE LINEAR FUNCTION GENERAL EQUATION

$y = mx + b$	Slope-intercept form.
$y - y_1 = m(x - x_1)$	Point-slope form. (x_1, y_1) is a point on the graph.
$Ax + By = C$	"$Ax + By = C$" form. A, B, and C stand for constants.

Objective:
Given an equation in point-slope form, plot the graph quickly, and transform it to the other two forms.

EXAMPLE

If $y - 5 = -\frac{3}{2}(x + 1)$,

a. Plot the graph quickly.
b. Transform the equation to slope-intercept form.
c. Transform the equation to $Ax + By = C$ form, where A, B, and C stand for *integer* constants.

Solution
a. From the equation, the slope is $-\frac{3}{2}$. A point on the graph is $(-1, 5)$ because substituting 5 for y makes the left member 0 and substituting -1 for x makes the right member 0. The graph is shown in Figure 3-3.

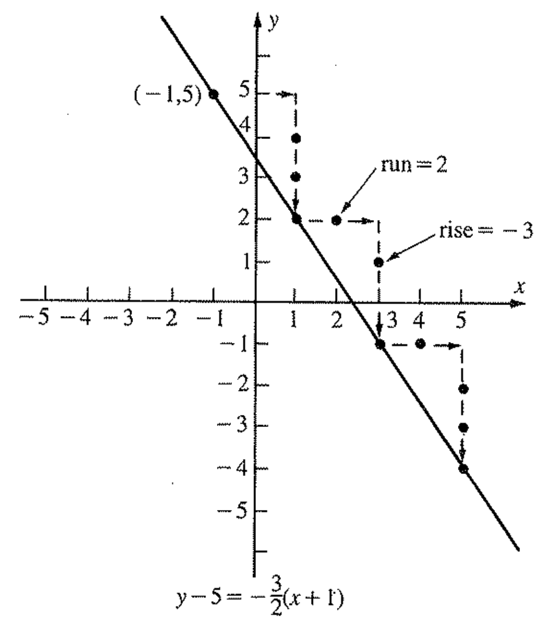


Figure 3-3

b. The equation can be transformed to slope-intercept form by distributing the $-\frac{3}{2}$, then adding 5 to each member.

$$y - 5 = -\frac{3}{2}(x + 1),$$

$$y - 5 = -1.5x - 1.5$$

$$\underline{\underline{y = -1.5x + 3.5}}$$

c. Starting with the answer in part (b), you can add $1.5x$ to each member, then multiply by 2 to make each coefficient an integer.

$$1.5x + y = 3.5$$

$$\underline{\underline{3x + 2y = 7}}$$

The following exercise gives you practice using the point-slope form to plot graphs, and transforming from point-slope form to the other forms.

EXERCISE 3-3

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Sketch a graph in which y increases as x increases.

Q2. Simplify: $3 + 2(x - 5)$

Q3. Multiply: $(2x - 7)(x + 3)$

Q4. Solve: $2x - 7 = 31$

Q5. Evaluate $5x^2$ if x is 3.

Q6. What percent of 40 is 12?

Q7. What axiom is illustrated: "$x = x$."

Q8. Evaluate: $\sqrt{49}$

Q9. Multiply $\frac{2}{3}$ by $\frac{3}{4}$.

Q10. Factor: $x^2 + 4x - 5$

Work these problems.

For Problems 1 through 10,

a. Plot the graph, showing clearly the point and slope that appear in the equation.
b. Transform the equation to slope-intercept form.
c. Transform the equation to $Ax + By = C$ form, where A, B, and C are all integers.

1. $y - 2 = \frac{3}{5}(x - 1)$
2. $y - 3 = \frac{2}{5}(x - 6)$
3. $y + 4 = \frac{7}{2}(x - 3)$
4. $y + 1 = \frac{7}{3}(x - 4)$
5. $y - 6 = -\frac{1}{4}(x + 2)$
6. $y - 2 = -\frac{1}{2}(x + 5)$
7. $y + 1 = -2(x + 4)$
8. $y + 6 = -3(x + 2)$
9. $y = \frac{1}{3}(x - 12)$
10. $y - 5 = \frac{2}{5}x$

If you really understand a concept, you should be able to use it *backward* as well as forward. For Problems 11 through 14, write an equation in point-slope form for the linear function described.

11. Contains the point (5, 7), and has slope −3.
12. Contains the point (6, 3), and has slope 5.
13. Contains the point (−2, 5), and has slope $\frac{9}{13}$.
14. Contains the point (7, −9), and has slope $-\frac{22}{7}$.

3-4 EQUATIONS OF LINEAR FUNCTIONS FROM THEIR GRAPHS

Suppose someone says, "If the equation is $y = 3x - 8$, what are the slope and y-intercept?" You would say, "That's easy! They are 3 and −8." It is just as easy for you to answer the question, "If the slope and y-intercept are −5 and 13, what is the equation?" The answer is

$$y = -5x + 13.$$

In this section you will use information about the graph to write equations of particular linear functions.

Objective:

Given information about the graph of a linear function, write its particular equation.

EXAMPLE 1

Find the particular equation of the linear function with slope $-\frac{3}{2}$, containing the point (7, −5).

Solution:

Since a point and the slope are given, the easiest form to use is the point-slope form. You would write

$$y + 5 = -\frac{3}{2}(x - 7)$$

The "+" is used on the left since the left member must be 0 when y is −5. The "−" is used on the right for similar reasons. It is not necessary to transform to any other form unless you are asked to do so. ■

EXAMPLE 2

Find the particular equation of the linear function containing the points $(-4, 5)$ and $(6, 10)$.

Solution:
This new problem can be turned into an old problem by first using the slope formula to find the slope, m.

$$m = \frac{10 - 5}{6 - (-4)} = \frac{5}{10} = 0.5$$

You can use either of the given points in the point-slope form.

$$\underline{y - 5 = 0.5(x + 4)} \quad \text{or} \quad \underline{y - 10 = 0.5(x - 6)}$$

These two equations are equivalent, as you can see by transforming each to slope-intercept form or $Ax + By = C$ form.

$$y = 0.5x + 7 \quad \text{or} \quad x - 2y = -14$$

Two lines are *parallel* to each other if their slopes are equal. Figure 3-4a illustrates this fact. If the lines are perpendicular to each other, the slope of one is the *opposite* of the *reciprocal* of the other. For instance, the slope of Line (2) in Figure 3-4a is $\frac{2}{3}$. The slope of Line (3), perpendicular to Line (2), is $-\frac{3}{2}$. These facts can be used to find particular equations.

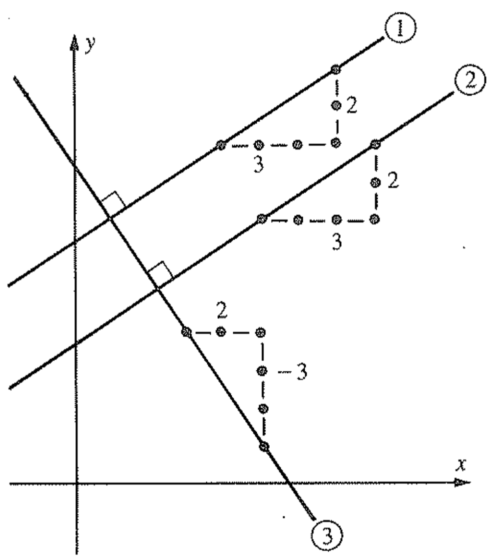


Figure 3-4a

PROPERTY

> **PARALLEL AND PERPENDICULAR LINES**
> If the equation of a line is $y = mx + b$, then:
>
> A **parallel** line also has slope m.
>
> A **perpendicular** line has slope $\frac{-1}{m}$.

EXAMPLE 3

Find the particular equation of the linear function containing $(-2, 7)$ if its graph is perpendicular to the graph of $3x + 4y = 72$.

Solution:
Transforming $3x + 4y = 72$ to $y = mx + b$ gives

$$y = -\frac{3}{4}x + 18$$

The slope of the given line is $-\frac{3}{4}$. So the slope of the desired line must be $\frac{4}{3}$, the opposite of the reciprocal of $-\frac{3}{4}$. The particular equation is thus

$$y - 7 = \frac{4}{3}(x + 2)$$

EXAMPLE 4

Find the particular equation of the horizontal line through $(7, 8)$.

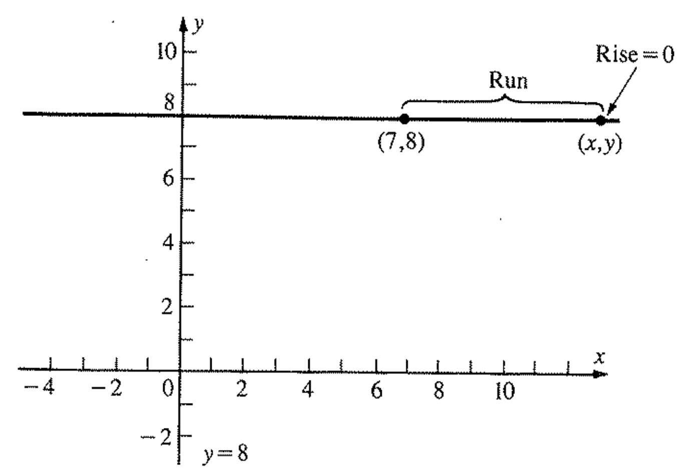


Figure 3-4b