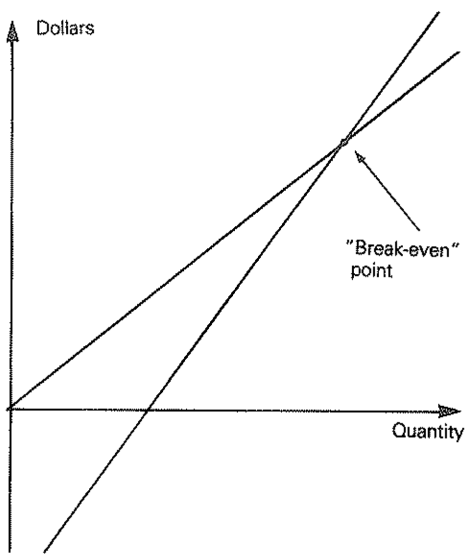
4

Systems of Linear Equations and Inequalities

*Linear functions may be used as mathematical models of the real world. Two or more linear functions with the **same** variables form what is called a **system** of equations. In this chapter you will learn how to find the intersection of the graphs, and how to tell what this intersection can represent. The techniques you develop can be used to find the "best" way to operate a business, such as raising race horses.*



For Problems T8 through T11, plot the graph.

T8. $y = -\frac{3}{5}x - 2$

T9. $y = 3x$

T10. $x - 4y = 12$

T11. $x = 2$

Some functions are not linear, and some relations are not functions.

T12. Plot the graph of the relation $|y| = x - 3$. Tell what the domain of the relation must be so that the value of y will be a real number. Tell whether or not the relation is a function.

4-1 INTRODUCTION TO LINEAR SYSTEMS

Two or more equations that have the same variables, such as

$$2x - y = 10$$
$$x + 3y = -9$$

form what is called a *system of equations*. A solution of such a system is an ordered pair that satisfies both of the equations at the same time. For this reason, the equations in a system are often called *simultaneous equations*. "Simultaneous" means "at the same time." Finding the solution is called "solving the system." By working the following exercise you will refresh your memory about how you solved systems in previous mathematics courses.

DEFINITIONS

SYSTEM
A **system** of equations or inequalities is a set of open sentences each of which contains the same variables.

SOLUTION SET
The **solution set** of a system with two variables is the set of all ordered pairs that satisfy all the open sentences in the system.

EXERCISE 4-1

These questions concern calculating the solution of the system

$$2x - y = 10 \text{ ---- (1)}$$
$$x + 3y = -9 \text{ ---- (2)}$$

1. Multiply each member of Equation (1) by 3. Label the answer ---- (3), as the given equations are labeled.
2. Add -9 to both sides of Equation (3). Do this by adding -9 to the right side, and $x + 3y$ (which equals -9) to the left side. The resulting equation should have only x as the variable. The y should have been eliminated.
3. Solve the equation from Problem 2 to find the value of x. This number will be the x-coordinate of the point where the two graphs cross each other.
4. To find the value of y at the point where the graphs cross, substitute the value of x you found in Problem 3 into either Equation (1) or Equation (2). Solve the resulting equation for y.
5. Write the ordered pair for the point where the graphs cross.
6. Plot the graphs of both equations on the same Cartesian coordinate system. Verify that the graphs really do cross at the point you wrote in Problem 5.

4-2 SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

In Exercise 4-1 you solved the system

$$\begin{aligned} 2x - y &= 10 \text{ ---- } (1) \\ x + 3y &= -9 \text{ ---- } (2) \end{aligned}$$

Multiplying each member of Equation (1) by 3 makes the y-coefficients opposites of each other.

$$\begin{aligned} 6x - 3y &= 30 \text{ ---- } (1) \text{ multiplied by } 3 \\ x + 3y &= -9 \text{ ---- } (2) \end{aligned}$$

Adding the two equations, left member to left member, and right to right, *eliminates* y, leaving an equation with only x as the variable.

$$7x \qquad = 21$$

Dividing each member by 7 gives

$$x = 3.$$

Substituting 3 for x in either Equation (1) or (2) gives an equation with only y as the variable.

$$\begin{aligned} 2(3) - y &= 10 && \text{Substitute 3 for } x \text{ in Eqn. (1).} \\ -y &= 4 \\ y &= -4 \end{aligned}$$

The two equations

$$x = 3$$
$$y = -4$$

form another system. Neither equation is equivalent to Equations (1) or (2). But the *systems* of equations *are* equivalent. As you can see from Figure 4-2a, both pairs of equations intersect at the same point, (3, −4). Obviously, the second system is simple enough to solve by inspection! You would write

$$S = \{(3, -4)\}$$

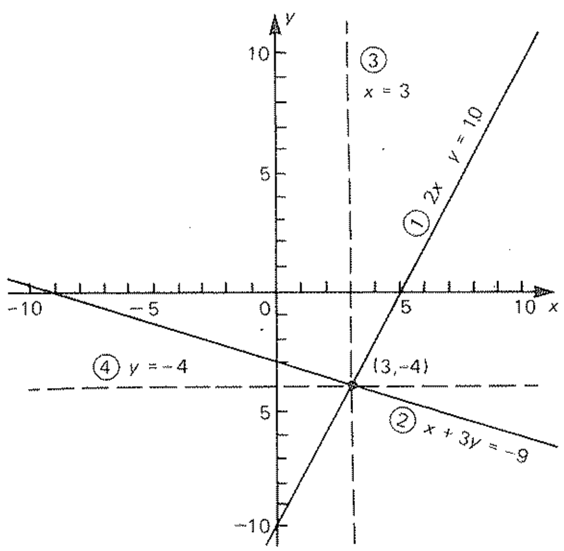


Figure 4-2a

Objective:

Given a system of two linear equations in two variables, solve it by transforming it to an equivalent system of the form

$$x = \text{constant},$$
$$y = \text{constant}.$$

The secret to eliminating a variable is making the x- or y-coefficients in the two equations either equal to each other or opposites of each other. The elimination can then be accomplished by subtracting the equations or by adding the equations.

EXAMPLE 1

Solve the system

$$\begin{aligned} 3x + 4y &= 6 \\ 5x - 7y &= 14 \end{aligned}$$

Solution:
Decide on which variable to eliminate. To eliminate x you could multiply the first equation by 5 and the second equation by 3, then subtract. A convenient way to keep track of what you do is to draw arrows to the right of the equations, and write messages such as "m5" over them to indicate multiplication by 5.

$$\begin{array}{rcl} 3x + 4y = 6 & \xrightarrow{\text{m5}} & 15x + 20y = 30 \\ 5x - 7y = 14 & \xrightarrow{\text{m3}} & \underline{15x - 21y = 42} \\ & & 41y = -12 \quad \text{Subtract.} \\ & & y = -0.2926... \quad \text{Divide by 41.} \end{array}$$

■

If the value of y turns out to be a decimal, you should write the first few digits, followed by an ellipsis mark, ... , to indicate that the more precise value is saved in your calculator's memory. Find x by substituting $-0.2926...$ for y in either equation.

$$\begin{aligned} 5x - 7(-0.2926...) &= 14 \quad \text{Substitute } -0.2926... \text{ for } y \text{ in Eqn. (2).} \\ 5x &= 11.95... \\ x &= 2.3902... \end{aligned}$$

The values of x and y can be checked by substituting into the *other* equation.

$$\begin{aligned} 3(2.3902...) + 4(-0.2926...) &= 6 \\ 6 &= 6, \text{ which checks.} \end{aligned}$$

The final answers may be rounded off to a reasonable number of places.

$$\underline{\underline{S = \{(2.39, -0.29)\}}}$$

Note that the answer is a solution set. The set contains the one ordered pair, (2.39, −0.29). So both the parentheses and the set symbols must be used.

The process of multiplying two variables by constants and adding the results is called "linearly combining" them. For example, $3r + 7s$ is a linear combination of r and s. This is why the name *linear combination method* is used for the process of solving a system of equations by multiplying each by a constant then adding or subtracting. The words *addition-subtraction method* are also used. In previous courses you have probably used the *substitution method* and the *graphing method*. Problems 41 through 50 in the following exercise will refresh your memory about those techniques.

EXAMPLE 2

Solve the system, and graph.

$$5x + 2y = 10$$
$$5x + 2y = 20$$

Solution:
Subtracting the equations gives

$$0 = -10.$$

This statement is never true, no matter what x or y may happen to be. As shown in Figure 4-2b, the graphs are parallel to each other, and thus do not intersect. So the solution set is empty and you would write

$S = \emptyset$. Equations are inconsistent.

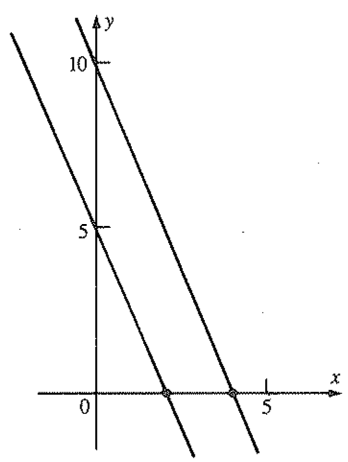
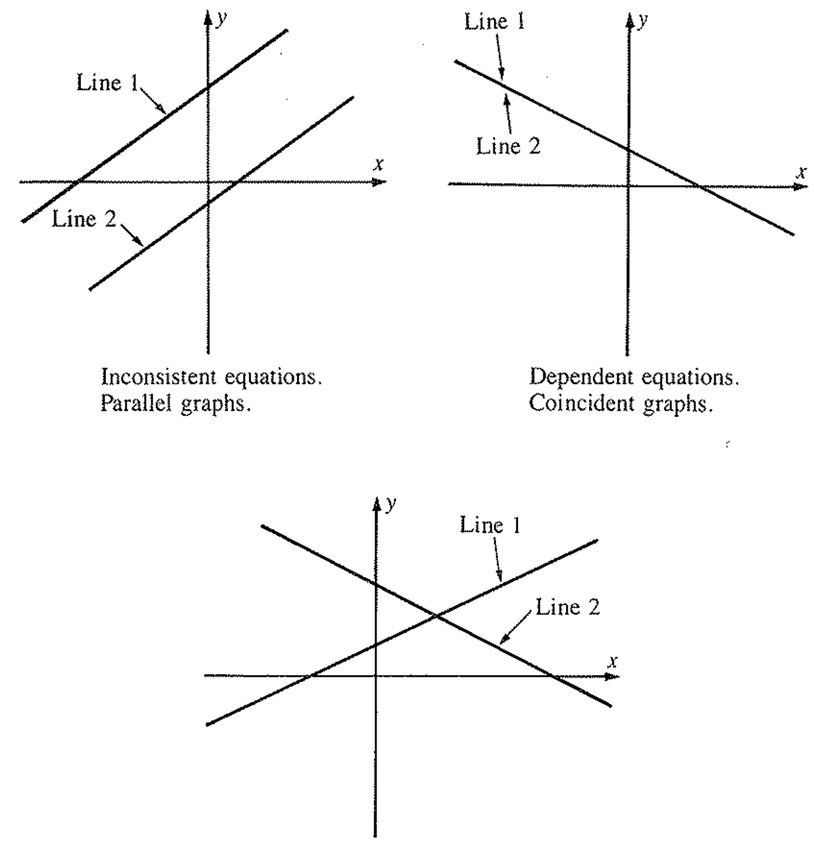


Figure 4-2b

The words *inconsistent equations* are used for any equations that have no common solutions. If the two equations in the system have the same graph, then they are called *dependent equations*. Such systems have an infinite number of solutions. Linear equations whose graphs intersect at just one point are called *independent equations*. Figure 4-2c illustrates the three possibilities.



Inconsistent equations.
Parallel graphs.

Dependent equations.
Coincident graphs.

Independent equations.
Intersecting graphs.

Figure 4-2c

EXERCISE 4-2

Do These Quickly

The following are 10 miscellaneous problems. They are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Find 30% of 90.

Q2. Subtract: 13.7 − 5

Q3. Find the probability of drawing a black marble from a bag containing 7 black marbles and 13 red ones.

Q4. Find the slope of the line connecting (2, −4) and (−3, 7).

Q5. Find the x-intercept: $y = 3x - 4$

Q6. Draw two parallel lines cut by a transversal, and indicate a pair of alternate interior angles.

Q7. Simplify: $32 - 2(3 + 4x)$

Q8. Write an imaginary number.

Q9. Write the equation that appears in the Associative Axiom for Addition.

Q10. Sketch the graph of a relation that is not a function.

For Problems 1 through 30, solve the system. Some of the later problems have additional instructions or suggestions.

1. $\begin{aligned} 5x + 2y &= 11 \\ x + y &= 4 \end{aligned}$

2. $\begin{aligned} x - y &= -11 \\ 7x + 4y &= -22 \end{aligned}$

3. $\begin{aligned} 6x - 7y &= 47 \\ 2x + 5y &= -21 \end{aligned}$

4. $\begin{aligned} 8x + 3y &= 41 \\ 6x + 5y &= 39 \end{aligned}$

5. $\begin{aligned} 5x + 7y &= -15 \\ 2x + 9y &= -6 \end{aligned}$

6. $\begin{aligned} 9x - 5y &= 26 \\ 4x - 3y &= 17 \end{aligned}$

7. $\begin{aligned} 2x + 3y &= 2 \\ 4x - 9y &= -1 \end{aligned}$

8. $\begin{aligned} 3x + 10y &= -24 \\ 6x + 7y &= -9 \end{aligned}$

9. $\begin{aligned} 9x - 7y &= 5 \\ 10x + 3y &= -16 \end{aligned}$

10. $\begin{aligned} 11x - 5y &= -38 \\ 9x + 2y &= -25 \end{aligned}$

For Problems 11 through 20, there is an *easy* multiplication that will work. In Problem 11, for example, x can be eliminated by multiplying the first equation by 3 and the second one by −2.

11. $\begin{aligned} 14x + 31y &= -6 \\ 21x + 17y &= 50 \end{aligned}$

12. $\begin{aligned} 12x - 7y &= 59 \\ 8x + 11y &= -39 \end{aligned}$

13. $\begin{aligned} 13x + 10y &= -7 \\ 17x - 15y &= 47 \end{aligned}$

14. $\begin{aligned} 13x + 20y &= -1 \\ 19x - 15y &= 87 \end{aligned}$

15. $\begin{aligned} 22x - 19y &= 28 \\ 55x - 29y &= 107 \end{aligned}$

16. $\begin{aligned} 37x - 36y &= 180 \\ 41x - 24y &= 120 \end{aligned}$

17. $\begin{aligned} -37x - 24y &= 35 \\ 15x - 18y &= 69 \end{aligned}$

18. $\begin{aligned} 18x - 19y &= 161 \\ 54x + 25y &= 565 \end{aligned}$

19. $\begin{aligned} 23x - 34y &= -46 \\ 13x + 51y &= -26 \end{aligned}$

20. $\begin{aligned} 33x + 16y &= -2 \\ -29x + 20y &= 138 \end{aligned}$

Problems 21 through 26 have "untidy" fractions for answers. You may want to use the linear combination to find *both* variables, rather than substituting a fraction back into one of the equations.

21. $4x - 3y = 11$
    $5x - 6y = 9$

22. $3x + 4y = 18$
    $9x + 6y = 17$

23. $3x + 4y = 8$
    $2x - 2y = 7$

24. $5x - 3y = 22$
    $6x - 7y = 41$

25. $-x + 5y = 22$
    $7x - 2y = 19$

26. $8x + 5y = 23$
    $3x - 2y = 37$

For Problems 27 through 30, you can write $\frac{2}{x}$ as $2\left(\frac{1}{x}\right)$, $\frac{-5}{y}$ as $-5\left(\frac{1}{y}\right)$, and so forth. Then you can solve for $\frac{1}{x}$ and $\frac{1}{y}$. To find x and y, recall that if two numbers are equal, then their reciprocals are equal.

27. $\frac{2}{x} - \frac{5}{y} = 5$
    $\frac{3}{x} + \frac{10}{y} = 18$

28. $\frac{3}{x} + \frac{6}{y} = 1$
    $\frac{3}{x} + \frac{7}{y} = 2$

29. $\frac{12}{x} + \frac{5}{y} = 25$
    $\frac{2}{x} - \frac{15}{y} = -18$

30. $\frac{6}{x} - \frac{7}{y} = 8$
    $\frac{15}{x} - \frac{14}{y} = 21$

For Problems 31 through 38, determine whether the equations in the system are independent, dependent, or inconsistent.

31. $15x + 12y = 8$
    $10x + 8y = 13$

32. $24x - 56y = 72$
    $-15x + 35y = -45$

33. $15x - 12y = 8$
    $10x + 8y = 13$

34. $24x - 56y = 72$
    $15x - 35y = -45$

35. $15x - 12y = 18$
    $10x - 8y = 12$

36. $24x + 56y = 72$
    $15x - 35y = -45$

37. $15x - 12y = 18$
    $10x - 8y = 14$

38. $24x - 56y = 72$
    $15x - 35y = 54$

For Problems 39 and 40, plot the three graphs on the same Cartesian coordinate system. They should cross at or near the same point. Then either find an ordered pair that satisfies all three equations, or demonstrate that there is no such ordered pair.

39. $5x + 3y = 19$
$2x - y = 9$
$4x - 3y = 19$

40. $3x - y = 5$
$2x + 3y = 7$
$x - 4y = 9$

Graphical Solution Method: For Problems 41 and 42, plot the graphs accurately by finding the x- and y-intercepts of each line. Read the coordinates of the intersection point to the nearest 0.1 unit. Then solve the system by linear combination and verify that your graphical solution is correct.

41. $2x + 3y = -12$
$8x - 5y = 40$

42. $5x + 2y = 20$
$3x - 7y = 21$

Substitution Method: It is possible to eliminate a variable by solving one equation for x in terms of y (or y in terms of x), and substituting the result into the other equation. For instance, in the Problem 43, below, the first equation can be transformed to $x = 2 - 2y$. Substituting $2 - 2y$ for x in the second equation eliminates x, giving $5(2 - 2y) - 3y = -29$. This equation can then be solved for y. For Problems 43 through 50, solve the system using this substitution method.

43. $x + 2y = 2$
$5x - 3y = -29$

44. $3x + y = 13$
$2x - 4y = 18$

45. $2x - 9y = 14$
$6x - y = 42$

46. $7x - 6y = -30$
$x - 4y = -20$

47. $x - 3y = 13$
$5x + 3y = 2$

48. $7x - 3y = -23$
$x - 5y = 32$

49. $x = 0.6(300 + y)$
$y = 0.2(300 + x)$

50. $x = 0.3(200 - y)$
$y = 0.2(200 - x)$

51. *Addition of Equations. Theory Problem:*
   a. For the system

$$3x + 5y = 17$$
$$4x - 5y = 23$$

   you could add the equations and eliminate y. The addition property of equality allows you to add the same number to each member of the first equation. Explain how this property and certain others let you add the two equations, left member to left member, and right member to right.
   b. Do a formal proof of the theorem which states, "If $p = q$ and $r = s$, then $p + r = q + s$."

4-3 SECOND-ORDER DETERMINANTS

Solving a system of equations is fairly tedious, especially if the coefficients are large numbers. If you solve a *general* system of linear equations, a pattern appears that is easy to remember. With the pattern, you can solve a system mentally, writing only the answer, if the coefficients are small enough, and by calculator otherwise.

The general system of linear equations with two variables is

$$\begin{aligned} ax + by &= c \\ dx + ey &= f, \end{aligned}$$

where a, b, c, d, e, and f stand for constant coefficients. To eliminate y, you would multiply the first equation by e and the second by b, then subtract.

$$\begin{array}{lcl} ax + by = c & \xrightarrow{\text{me}} & aex + bey = ce \\ dx + ey = f & \xrightarrow{\text{mb}} & \underline{bdx + bey = bf} \\ & & aex - bdx = ce - bf \quad \text{Subtract.} \end{array}$$

The x can be factored from the left member, giving

$$(ae - bd)x = ce - bf$$

The result is an equation whose form is as simple as $3x = 7$. You merely divide each member by $(ae - bd)$, the coefficient of x.

$$x = \frac{ce - bf}{ae - bd}$$

This is a formula that can be used to evaluate x. Eliminating x and solving the original system for y gives

$$y = \frac{af - cd}{ae - bd}$$

Three things appear when you compare these formulas with the original system,

$$\begin{aligned} ax + by &= c \\ dx + ey &= f. \end{aligned}$$

1. Both denominators are the *same*, and contain only the coefficients of the *left* members of the equations.
2. The numerator for x does *not* contain the x-coefficients, a and d.
3. The numerator for y does *not* contain the y-coefficients, b and e.

There is a way to remember how to get the denominators. You write the four coefficients of x and y this way:

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix}.$$

The value is found by a diagonal multiplication scheme. Multiply *top left* by *bottom right*. Subtract *top right* times *bottom left*.

$$D = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$$

Looking back at the formulas for x and y, you can see that this is actually the correct denominator.

The four-number symbol with vertical bars is called a *second-order determinant*.

DEFINITION

SECOND-ORDER DETERMINANT
A **second-order determinant** is a square array of numbers that is expanded (evaluated) according to the rule

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = ru - st$$

The letter D is used for Denominator determinant. The numerators in the formulas can also be written as determinants. Recalling that the x-numerator does not have the coefficients of x, you simply replace the x-coefficients in the left members of the equations with the constants c and f from the right members.

Delete a and d.

$$ax + by = c$$
$$dx + ey = f$$

Replace with c and f.

You get a new determinant with c and f in the x-column (and b and e still in the y-column).

$$N_x = \begin{vmatrix} c & b \\ f & e \end{vmatrix}$$

The symbol N_x stands for "x-numerator." Expanding this determinant by the diagonal multiplication pattern gives

$$N_x = \begin{vmatrix} c & b \\ f & e \end{vmatrix} = ce - bf,$$

which is the correct value. The y-numerator, N_y, is found by replacing the y-coefficients, b and e, with the constant terms c and f.

Delete b and e.

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

Replace with c and f.

$$N_y = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - cd,$$

which is the correct y-numerator. The solutions are N_x/D and N_y/D

Objective:

Given a system of two linear equations with two variables, solve the system using second-order determinants.

Solving by determinants is often called *Cramer's Rule.*

EXAMPLE

Solve by determinants (Cramer's Rule):

$$\begin{aligned} 3x + 4y &= 2 \\ 5x - 7y &= 17 \end{aligned}$$

Solution:
The first step is to write the denominator for x.

$$x = \frac{}{\begin{vmatrix} 3 & 4 \\ 5 & -7 \end{vmatrix}}$$

Next, you write the numerator determinant by replacing the numbers in the x-column with the 2 and 17 from the right members of the equations.

$$x = \frac{\begin{vmatrix} 2 & 4 \\ 17 & -7 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & -7 \end{vmatrix}}$$

Then you expand the two determinants.

$$x = \frac{\begin{vmatrix} 2 & 4 \\ 17 & -7 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & -7 \end{vmatrix}} = \frac{-14 - 68}{-21 - 20} = \frac{-82}{-41} = 2,$$

For y, you write the numerator determinant by replacing the numbers in the y-column of the *denominator* determinant with the 2 and 17 from the right members of the equations. You already know that the denominator is -41, so you write

$$y = \frac{\begin{vmatrix} 3 & 2 \\ 5 & 17 \end{vmatrix}}{-41} = \frac{51 - 10}{-41} = \frac{41}{-41} = -1.$$

$$\therefore S = \{(2, -1)\}.$$

In the following exercise you will solve systems by Cramer's rule.

EXERCISE 4-3

Do These Quickly

The following are 10 miscellaneous problems. They are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Solve by linear combination: $3x + y = 11$ and $x + y = 5$

Q2. Subtract: $\frac{2}{3} - \frac{1}{6}$

Q3. What is the probability of getting a 3 or larger in one roll of a die?

Q4. Sketch the graph of a line with infinite slope.

Q5. Evaluate 2^5.

Q6. Draw a rhombus.

Q7. Simplify: $37 - 7(3 - 4x)$

Q8. Write a transcendental number.

Q9. Write the equation that appears in the Reflexive Axiom.

Q10. If 40% of a class of 30 students are male, how many are female?

For Problems 1 through 16, solve the system using second-order determinants. Problems 1 through 10 are the same as Problems 1 through 10 in Exercise 4-2. Problems 11 through 16 are the same as Problems 21 through 26 in Exercise 4-2.

1. $\begin{aligned} 5x + 2y &= 11 \\ x + y &= 4 \end{aligned}$

2. $\begin{aligned} x - y &= -11 \\ 7x + 4y &= -22 \end{aligned}$

3. $\begin{aligned} 6x - 7y &= 47 \\ 2x + 5y &= -21 \end{aligned}$

4. $\begin{aligned} 8x + 3y &= 41 \\ 6x + 5y &= 39 \end{aligned}$

5. $\begin{aligned} 5x + 7y &= -15 \\ 2x + 9y &= -6 \end{aligned}$

6. $\begin{aligned} 9x - 5y &= 26 \\ 4x - 3y &= 17 \end{aligned}$

7. $\begin{aligned} 2x + 3y &= 2 \\ 4x - 9y &= -1 \end{aligned}$

8. $\begin{aligned} 3x + 10y &= -24 \\ 6x + 7y &= -9 \end{aligned}$

9. $\begin{aligned} 9x - 7y &= 5 \\ 10x + 3y &= -16 \end{aligned}$

10. $\begin{aligned} 11x - 5y &= -38 \\ 9x + 2y &= -25 \end{aligned}$

11. $\begin{aligned} 4x - 3y &= 11 \\ 5x - 6y &= 9 \end{aligned}$

12. $\begin{aligned} 3x + 4y &= 18 \\ 9x + 6y &= 17 \end{aligned}$

13. $\begin{aligned} 3x + 4y &= 8 \\ 2x - 2y &= 7 \end{aligned}$

14. $\begin{aligned} 5x - 3y &= 22 \\ 6x - 7y &= 41 \end{aligned}$

15. $\begin{aligned} -x + 5y &= 22 \\ 7x - 2y &= 19 \end{aligned}$

16. $\begin{aligned} 8x + 5y &= 23 \\ 3x - 2y &= 37 \end{aligned}$

17. a. Graph the following system of equations and write the ordered pair at which they seem to intersect.

$$\begin{aligned} 5x - 2y &= 4 \\ 3x + 7y &= 26 \end{aligned}$$

b. Solve the above system using determinants and express the answers as mixed numbers.

c. Based on the answers to parts a and b, why do you suppose that it is safer to get answers by *calculation* than it is by graphing?

18. a. Plot the graph of each of the following equations on the *same* Cartesian coordinate system:

$$\begin{aligned} x + y &= 5 \quad \text{——} \quad ① \\ 3x - 2y &= 8 \quad \text{——} \quad ② \\ x + 3y &= 8 \quad \text{——} \quad ③ \end{aligned}$$

If you have done the work correctly, the graphs will all intersect at (or near) the *same* point.

b. Solve the systems formed by equation ① and ②, by ② and ③, and by ① and ③ using determinants. Write the answers as mixed numbers and by comparing the three answers, tell whether or not all three *really* intersect at the same point.

19. ***Inconsistent Equations, Dependent Equations, and Determinants***
a. On *separate* sets of axes, plot the graphs of the following two systems:

i. $3x + 2y = 7$
$6x + 4y = 8$

ii. $3x + 2y = 7$
$6x + 4y = 14$

b. From your graphs, tell which pair of equations is *inconsistent* (graphs are parallel), and which pair of equations is *dependent* (graphs coincide).
c. Show that the denominator determinant for each system equals zero.
d. Find the numerator determinant, N_x, for each system. How can you tell from the value of this determinant whether the equations are inconsistent or dependent?
e. Solve the following systems by determinants. If the denominator determinant equals zero, tell whether the equations are inconsistent or dependent.

i. $15x + 12y = 8$
$10x + 8y = 13$

ii. $15x - 12y = 8$
$10x + 8y = 13$

iii. $15x - 12y = 18$
$10x - 8y = 12$

iv. $15x - 12y = 18$
$10x - 8y = 14$

20. *Computer Solution of Linear Systems* Write a computer program that will solve a system of two linear equations with two variables. The input should be the six coefficients, three for one equation and three for the other. The computer should evaluate the denominator and the two numerators, and print these values, along with messages to tell which is which. If the denominator is zero, the program should print the message "INCONSISTENT" or "DEPENDENT," whichever is correct. Otherwise, the computer should do the dividing and print the answer as an ordered pair. Debug your program by solving all four systems in Problem 19, part (e), above.

21. *Computer Graphics for Linear Systems* The solution of a system of linear equations is the ordered pair where the graphs cross each other. This ordered pair can be found by actually graphing the two equations and reading the ordered pair. An efficient way to do the graphing is by computer. For each of the systems below, use the program PLOT LINEAR on the accompanying disk, or similar plotting program, to draw the two graphs. You may need to change scales or axis locations to make the intersection appear on the screen. Then estimate the coordinates of the intersection point to one decimal place. The GRID and POINT options will help. In each case, show

by calculation that the ordered pair you get really does satisfy both equations in the system.

a. $9x - 7y = 5$
$10x + 3y = -16$

b. $8x + 5y = 23$
$3x - 2y = 37$

22. ***Mental Solution of Systems*** With practice, you can solve simple systems of linears in your head, writing down only the answer. The secret is being able to visualize the diagonal multiplication scheme, without actually writing the determinants. Solve the following systems mentally. Then check the answers in the back of the book to make sure you are right. If not, try the problem again, mentally, till you get it right. The solutions are not necessarily integers.

a. $2x + 3y = 8$
$5x + y = 6$

b. $4x - 3y = 1$
$6x + 2y = -5$

23. ***Formula Derivation Problem*** Given the general system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f, \end{aligned}$$

solve for y by linear combination, eliminating y. Show that your answer is equivalent to the formula in the text.

4-4 f(x) TERMINOLOGY, AND SYSTEMS AS MODELS

In the preceding sections you have studied systems of equations such as

$$\begin{aligned} 3x - 4y &= -12 \\ 5x - 2y &= 10. \end{aligned}$$

For graphing purposes, each equation could be transformed to point-slope form, $y = mx + b$, giving

$$y = \frac{3}{4}x + 3$$

$$y = \frac{5}{2}x - 5.$$

The graphs are shown in Figure 4-4a.

The equations can be thought of as representing two different functions with the same independent variable, x. For example, a newspaper carrier's expenses and income both depend on the number of papers he or she delivers. The intersection point of the two graphs would represent the number of papers for which the income equals the expenses (the "break-even" point).

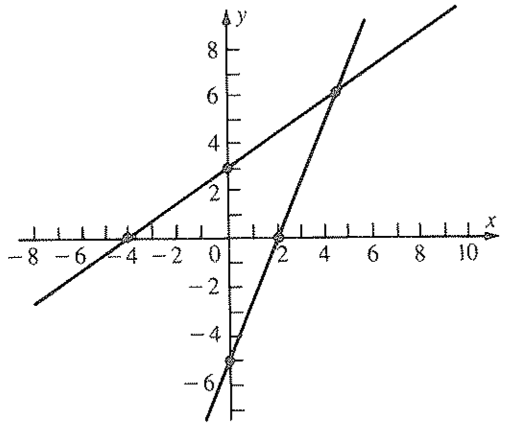


Figure 4-4a

Suppose someone asks, "What does y equal when x is 4?" You cannot answer this question since there are two different functions. Fortunately, mathematicians use terminology that allows you to distinguish between functions with the same independent variable. They write

$$f(x) = \frac{3}{4}x + 3$$

$$g(x) = \frac{5}{2}x - 5.$$

The symbol $f(x)$ is pronounced "f of x." It means "the value of y in function f, where the independent variable is x." Similarly, $g(x)$ means the value of y in another function, g, with independent variable x. Any letter may be used for the name of a function. The letters f and g are often used because f is the first letter of the "function," and g comes next to f in the alphabet.

This "$f(x)$ terminology" allows you to show what value is being substituted for x. For instance, the symbol $f(4)$ would mean the value of y in function f when 4 is substituted for x. The symbol $g(4)$ would have the same meaning for function g.

$$f(4) = \frac{3}{4} \cdot 4 + 3 = 3 + 3 = 6,$$

$$g(4) = \frac{5}{2} \cdot 4 - 5 = 10 - 5 = 5.$$

DEFINITION

> ***f(x)***
> The symbol $f(x)$, pronounced "f of x," or "f at x," means the value of the dependent variable in function f if the independent variable is x.

Beware! Do not misinterpret $f(x)$ as meaning f times x. The letter f is simply the *name* of the function, not a variable.

Objectives:

1. Become comfortable using $f(x)$ terminology by using it to evaluate functions.
2. Use systems of linear functions as mathematical models.

EXAMPLE 1

Given $f(x) = 3x^2$ and $g(x) = 4x + 1$, find

a. f(5) b. $g(5)$ c. $\dfrac{f(2)}{g(2)}$ d. $f(g(6))$

Solutions:

a. $f(5) = 3 \cdot 5^2 = 3 \cdot 25 = \underline{\underline{75}}$
b. $g(5) = 4 \cdot 5 + 1 = \underline{\underline{21}}$
c. $\dfrac{f(2)}{g(2)} = \dfrac{3 \cdot 2^2}{4 \cdot 2 + 1} = \underline{\underline{\dfrac{12}{9}}}$

Note that you can *not* cancel the 2's in $\dfrac{f(2)}{g(2)}$ because $f(2)$ and $g(2)$ are simply *names* for the y-values in the functions.

d. $f(g(6)) = f(4 \cdot 6 + 1) = f(25) = 3 \cdot 25^2 = \underline{\underline{1875}}$

The symbol $f(g(6))$ is pronounced, "f of g of 6." As is usual in algebra, you work what is inside the parentheses first. ■

EXAMPLE 2

Air Conditioner Problem Suppose that your family is going to buy a new air conditioning unit. One brand costs \$1800 to buy, and \$60 a month to operate. A more expensive band costs \$2600 to buy. But it is more efficient, and costs only \$50 a month to operate.

Let x = number of months since you purchased the unit.

Let $f(x)$ = total number of dollars spent in x months if you buy the cheaper unit.
Let $g(x)$ = total number of dollars spent in x months if you buy the more expensive unit.

a. Write the particular equations expressing $f(x)$ and $g(x)$ in terms of x.
b. Find $f(100)$ and $g(100)$. What do these numbers tell you about the relative costs of the two units after 100 months?
c. Plot the graphs of functions f and g on the same set of axes.
d. Solve the system of equations in part (a) to find the "break-even" point. That is, find the number of months for which the total cost of either unit would be the same.

Solution:

a. Since it costs \$60 a month to run the cheaper unit, it will cost $60x$ dollars to run it for x months. So the total cost is

$$f(x) = 60x + 1800.$$

Similarly, the cost of the more expensive unit is

$$g(x) = 50x + 2600.$$

b. $f(100) = 60 \cdot 100 + 1800 = \underline{7800}$
$g(100) = 50 \cdot 100 + 2600 = \underline{7600}$

So it costs *less* to own the *expensive* (efficient) unit if you keep it as long as 100 months!

c. The "y"-intercepts are $f(0) = 1800$ and $g(0) = 2600$. Since $f(100) = 7800$, the point (100, 7800) is on the f graph. Similarly, (100, 7600) is on the g graph. The result is as in Figure 4-4b.

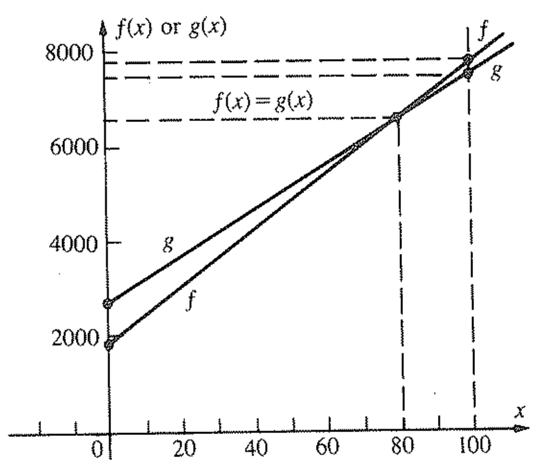


Figure 4-4b

d. The easiest way to solve the system in part (a) is to realize that where the graphs cross, $f(x) = g(x)$.

$$\begin{aligned} 60x + 1800 &= 50x + 2600 \quad \text{Equate } f(x) \text{ and } g(x). \\ 10x &= 800 \\ x &= 80 \end{aligned}$$

So you break even after 80 months.

The following exercise gives you some practice using $f(x)$ terminology.

EXERCISE 4-4

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Find 40% of 2.7.

Q2. Find the slope of the line from $(-3, 5)$ to $(4, 8)$.

Q3. Solve for p: $\dfrac{3}{p} = \dfrac{5}{7}$

Q4. Solve for j: $|j - 2| = 5$

Q5. Commute the 2 and the $3x$: $5 + 3x + 2$

Q6. Evaluate the determinant: $\begin{vmatrix} 4 & 7 \\ 2 & 9 \end{vmatrix}$

Q7. Write the numerator determinant for y: $\begin{aligned} 3x + 7y &= 4 \\ 5x - 6y &= 8 \end{aligned}$

Q8. Draw a trapezoid.

Q9. Sketch the graph of a function that is not linear.

Q10. Solve for x: $-5x > 20$

For Problems 1 through 26, let

$$\begin{aligned} f(x) &= 3x + 11 \\ g(x) &= x^2 + x + 1 \end{aligned}$$

Evaluating the following:

1. $f(7)$
2. $g(5)$
3. $g(-3)$
4. $f(-4)$