5. $f(0)$
6. $g(0)$
7. $g\left(\frac{1}{3}\right)$
8. $f(0.5)$
9. $\frac{f(5)}{g(5)}$
10. $\frac{g(2)}{f(2)}$
11. $\frac{g(1)}{g(0)}$
12. $\frac{f(6)}{g(3)}$
13. $f(g(2))$
14. $g(f(-5))$
15. $g(g(0))$
16. $f(f(-2))$

For Problems 17 through 26, you should realize that f ("expression") means substitute "expression" for x. Evaluate, and simplify if possible.

17. $f(r)$
18. $g(n)$
19. $g(k)$
20. $f(j)$
21. $f(s + t)$
22. $g(4 - a)$
23. $g(f(x))$
24. $f(g(x))$
25. $f(f(x))$
26. $g(g(x))$
27. ***Cops and Robbers Problem*** Robin Banks robs a bank and drives off. A short time later he passes a truck stop at which police officer Willie Katchup is dining. Willie receives a call from his dispatcher, and takes off in pursuit of Robin.

    Let t = number of minutes that have elapsed since Robin passed the truck stop.
    Let $f(t)$ = number of kilometers Robin has gone past the truck stop.
    Let $g(t)$ = number of kilometers Willie has gone from the truck stop.

    a. Robin's equation is $f(t) = 0.75t$. Find $f(12)$, $f(4)$, and $f(-8)$.
    b. Willie's equation is $g(t) = 2(t - 5)$. Find $g(7)$ and $g(15)$.
    c. By calculation, find the time and place Willie Katchup catches up with Robin Banks.
    d. When did Willie leave the truck stop?
    e. Sketch the graphs of functions f and g on the same set of axes, showing the point where they cross.
    f. How fast were Robin and Willie going?
28. ***Pedalboat Problem*** Eb and Flo go pedalboating on the San Antonio River. They check out a boat, head out along the river for awhile, then turn around and come back to the boatdock.

Let t = number of minutes that have elapsed since they left the boatdock.
Let $f(t)$ = number of meters they are from the boatdock on the way out.
Let $g(t)$ = number of meters they are from the boatdock on the way back in.

a. They find that the equation for function f is $f(t) = 32t$. Find $f(3)$, $f(7)$, and $f(10)$.
b. The equation for function g is $g(t) = -17t + 510$. Find $g(18)$ and $g(23)$.
c. Sketch the graphs of f and g on the same set of axes. Use dotted lines for the graphs in parts of the domain where the equations do not apply.
d. Calculate the point where the graphs intersect.
e. What was happening in the real world at the time in part (d)?
f. When did they arrive back at the boatdock?
g. Assuming that they were going the same speed through the water for both parts of the trip, did they start out going upstream or downstream? Justify your answer.
h. Just for fun, see if you can figure out the speed of the current in the San Antonio River.

29. ***Efficient Car Problem*** A particular brand of car with the normal engine costs $11,000 to purchase, and 22 cents a mile to drive. The same car with a fuel-injection engine costs $11,300 to purchase, but only 20 cents a mile to operate.
    a. Let d be a variable equal to the number of miles you have driven the car, and $f(d)$ be the total number of dollars it costs to own the $11,000 car. Write the particular equation for function f.
    b. Calculate $f(1{,}000)$, $f(10{,}000)$, and $f(100{,}000)$.
    c. Let $g(d)$ be the total number of dollars it costs to drive the $11,300 car for d miles. Write the particular equation for function g.
    d. Calculate $g(1{,}000)$, $g(10{,}000)$, and $g(100{,}000)$.
    e. How many miles would you have to drive to "break even?" That is, when does the total cost of owning the car with the normal engine equal the cost of owning the car with the fuel-injected engine?

30. ***Hamburger Problem*** Sue Flay and Cassa Roll obtain a franchise to operate a hamburger stand for a well-known national hamburger chain. They pay $20,000 for the franchise, and have additional expenses of $250 per thousand hamburgers they sell. They sell the hamburgers for $.75 each, so they take in a revenue of $750 per thousand burgers.
    a. Let $r(x)$ be the number of dollars revenue they take in by selling x thousand burgers. Write the particular equation for function r.

b. Find $r(20)$, $r(50)$, and $r(0)$.
c. Let $c(x)$ be the total cost of owning the hamburger stand, including the $20,000 franchise fee. Write the particular equation for function c.
d. Find $c(20)$, $c(50)$, and $c(0)$.
e. Sketch the graphs of r and c on the same set of axes. Have they crossed by the time x is 50?
f. How many burgers must Sue and Cassa sell in order to break even?

4-5 LINEAR EQUATIONS WITH THREE OR MORE VARIABLES

In Section 4-4 you studied situations in which there were *three* variables. Two dependent variables such as distance were related to one independent variable such as time. In this section you will study linear equations with three variables *without* being concerned with which are dependent and which are independent.

Objective:
Determine what the graph of a linear equation with three variables looks like, and be able to sketch the graph.

Since you are not concerned with whether the variables are dependent or independent, you usually write equations with the variables on one side and the constant on the other. For example,

$$2x + 3y + 4z = 12.$$

A solution to such an equation must contain *three* numbers, one for each of the three variables. It is customary to write the solutions as ordered *triples;* for example (4, 0, 1). The order in which the numbers appear tells which variable they stand for. If the equation contained four variables, the solutions would be called ordered *quadruples;* for five variables, ordered *quintuples;* for n variables, ordered *n-tuples*.

Agreement: Unless otherwise specified, variables in ordered n-tuples will come in alphabetical order.

Equations with three variables can be graphed on a three-dimensional Cartesian coordinate system. In addition to the normal x- and y-axes, there is a z-axis perpendicular to the xy-plane, passing through the origin. The positive portions of the three axes are shown in Figure 4-5a.

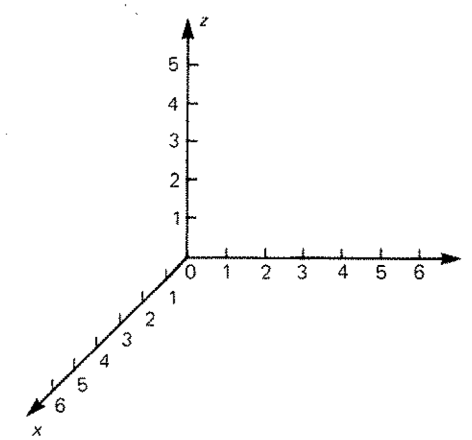


Figure 4-5a

Plotting an equation such as

$$2x + 3y + 4z = 12,$$

can be accomplished by picking values of *one* variable, and seeing what you get for the others.

If $z = 0$, then $2x + 3y = 12$ ______. ①

If $z = 1$, then $2x + 3y + 4 = 12$
$2x + 3y = 8$ ______. ②

If $z = 2$, then $2x + 3y + 8 = 12$
$2x + 3y = 4$ ______. ③

Graphs of Equations ①, ②, and ③ are shown in Figure 4-5b.

The three lines in Figure 4-5b are the parts of the graph at $z = 0$, at $z = 1$, and at $z = 2$. By stacking theses three planes on top of each other (Figure 4-5c), you can see that the whole graph is a *plane* in *space*, containing these three lines.

Conclusion: The graph of a linear equation with three variables is a *plane* in *space*.

Once you realize what the graph looks like, you can draw it more quickly. The line where the graph cuts the xy-plane is called the graph of the *xy-trace*. It is obtained by setting $z = 0$. Similarly, the yz-trace and xz-trace are found by letting x and y equal zero, respectively. By drawing these three traces, you get a reasonable picture of the plane (Figure 4-5d).

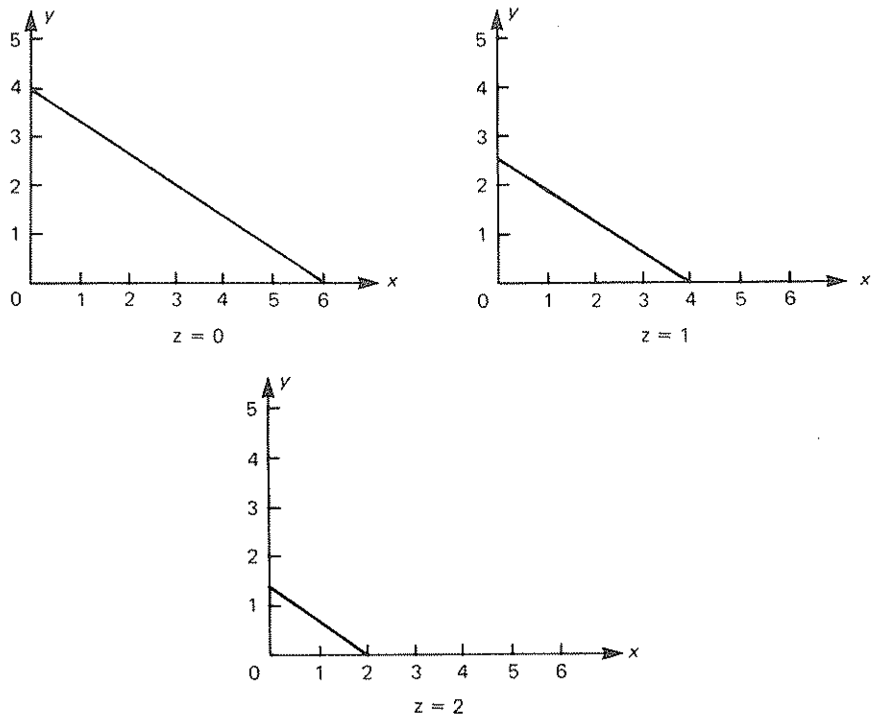


Figure 4-5b

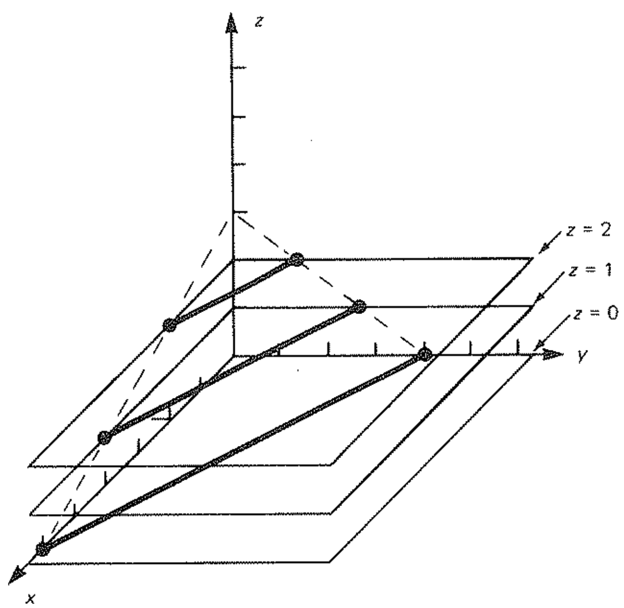


Figure 4-5c

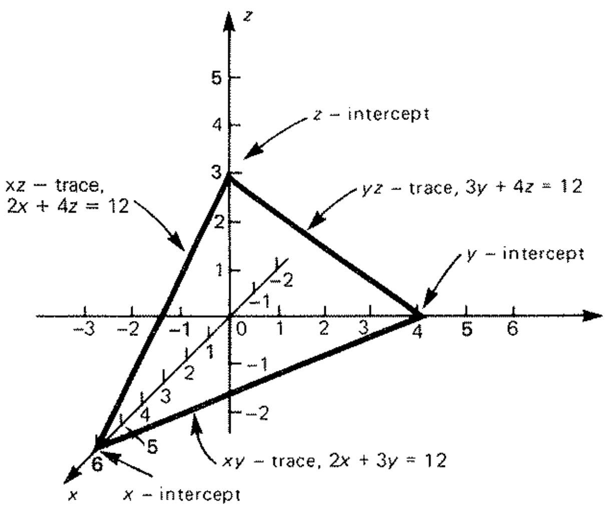


Figure 4-5d

DEFINITION

> **TRACE**
> The ***xy*-trace** is the set of ordered pairs (x, y) obtained by setting $z = 0$. The other traces are similarly defined.

Setting *two* variables equal to zero gives an *intercept*. For example, if y and z both equal zero, then

$$2x + 0 + 0 = 12,$$
$$x = 6.$$

So the x-intercept equals 6. Similarly, the y- and z-intercepts are 4 and 3, respectively as shown in Figure 4-5d.

DEFINITION

> **INTERCEPT**
> The ***x*-intercept** is the value of x when y and z are both zero. The other intercepts are similarly defined.

The idea of traces and intercepts can be extended to equations with more than three variables. Since you have used an *algebraic* definition of these quantities, the fact that there can be no four or five dimensional graphs becomes insignificant. A trace is obtained simply by letting *one* variable equal zero, while an intercept is obtained by setting *all but one* variable equal to zero.

EXERCISE 4-5

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Write the particular equation of the linear function containing (3, 7) and (5, 12).

Q2. Sketch the graph of a relation that is not a function.

Q3. Multiply 37.4925 by 1000 *without* a calculator!

Q4. Find 3% of 33.

Q5. Divide 50 by one-half, and add three.

Q6. Solve: $|x - 2| = -7$

Q7. Is $\sqrt{-25}$ a real number?

Q8. Name by degree and number of terms: $5y^2 - 4y + 7$

Q9. Factor: $x^2 + 2x - 35$

Q10. Draw a triangle inscribed in a circle.

Work the following problems.

1. Sketch a graph of each of the following equations by drawing its three traces as in Figure 4-5d.
   a. $6x + 4y + 3z = 24$
   b. $2x - 3y + z = 12$
   c. $3x + 5y - 3z = 15$
   d. $4x - 2y - z = 8$
   e. $x + y + z = -7$
2. There are some interesting special cases in which the graphs turn out to be parallel, perpendicular, or coincident with the coordinate planes or axes. Sketch a graph of each of the following equations by drawing their traces as in Problem 1. Then tell what the graph is parallel to, or coincident with.

a. $x + y = 7$
b. $y + z = 4$
c. $x + y = 0$
d. $y - z = 0$
e. $x = 5$
f. $y = -6$

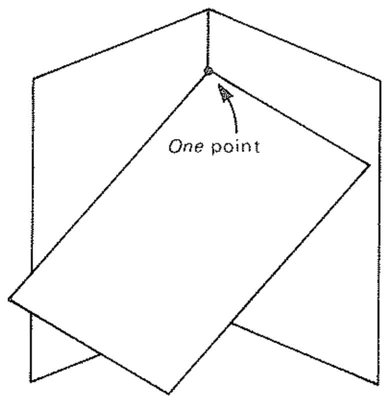
3. For the equation $7w + 3x - 4y + 6z = 42$,
   a. find the four intercepts, and
   b. find the equation of the xyz-trace.

4. Explain why a trace is the same as an intercept for an equation with two variables.

5. Obtain three pieces of cardboard; playing cards or index cards will do. These will represent graphs of equations with three variables.
   a. Hold two of the cardboards together so that they meet along an edge. What do the points of intersection represent with regard to a system of two equations in three variables? How many ordered triples normally satisfy a system of two equations in three variables?
   b. Hold the third cardboard so that its corner touches the line of intersection of the other two. Why do you suppose that a system of *three* linear equations in three variables has a *unique* solution, whereas a system with only two does *not*?
   c. Hold the three cardboards in such a way that they intersect at an *infinite* number of points. There are at least two ways to do this. If this happens, the three equations are said to be "dependent."
   d. Hold the three cardboards in such a way that there are *no* points common to all three. There are at least three ways of doing this besides the obvious way of three parallel planes. If this happens, the three equations are said to be "inconsistent."
   e. From what you observed above, can you conclude that a system of three linear equations in three variables *always* has a unique solution? Explain.

4-6 SYSTEMS OF LINEAR EQUATIONS WITH THREE OR MORE VARIABLES

The graph of a linear equation with three variables is a plane in space. If you hold three index cards as shown in Figure 4-6, you can see that *three* such planes usually intersect at a *single* point. So a system of three linear equations with three variables will usually have a *single* ordered triple in its solution set.



Three intersecting planes

Figure 4-6

Objective:

Be able to find the single ordered triple that satisfies a system of three linear equations with three variables.

EXAMPLE

Solve the system

$$\begin{aligned} 2x + 3y - z &= -1 \\ -x + 5y + 3z &= -10 \\ 3x - y - 6z &= 5 \end{aligned}$$

Solution:

The technique is to *eliminate* a variable, and get a system of two equations with two variables. Suppose you choose to eliminate x by linearly combining the first and second equations. The work would look like this:

$$2x + 3y - z = -1 \xrightarrow{\text{m1}} 2x + 3y - z = -1 \searrow$$

$$13y + 5z = -21$$

$$-x + 5y + 3z = -10 \xrightarrow{\text{m2}} -2x + 10y + 6z = -20 \nearrow$$

The equation $13y + 5z = -21$ has only y and z. To get another equation with y and z you can eliminate x by linearly combining a different pair of equations. Using the second and third gives

$$-x + 5y + 3z = -10 \xrightarrow{\text{m3}} -3x + 15y + 9z = -30 \searrow$$

$$14y + 3z = -25$$

$$3x - y - 6z = 5 \xrightarrow{\text{m1}} 3x - y - 6z = 5 \nearrow$$

From here on it is an old problem. These two equations can be linearly combined to eliminate z as follows:

$$13y + 5z = -21 \xrightarrow{\text{m3}} 39y + 15z = -63$$

$$14y + 3z = -25 \xrightarrow{\text{m5}} 70y + 15z = -125$$

$$\longrightarrow 31y = -62$$

Dividing by 31 gives

$$y = -2.$$

Substituting -2 for y in one of the two-variable equations gives

$$\begin{aligned} -26 + 5z &= -21 \\ 5z &= 5 \\ z &= 1. \end{aligned}$$

Substituting -2 for y and 1 for z in one of the original equations gives

$$\begin{aligned} 2x - 7 &= -1 \\ 2x &= 6 \\ x &= 3 \\ \therefore S &= \{(3, -2, 1)\} \end{aligned}$$

There is quite a bit of work involved in solving a three-variable system. The horizontal format, shown above, seems to make the work as easy as is possible.

In the following exercise you will practice solving systems with three variables. You will also apply what you have learned to systems with more than three variables.

EXERCISE 4-6

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Sketch the plane with intercepts $x = 5$, $y = 3$, and $z = 8$.

Q2. Find the slope of $3x - 7y = 42$.

Q3. If $f(x) = 3x^2$, find $f(-4)$.

Q4. Show two corresponding angles if two parallel lines are cut by a transversal.

Q5. Write the hypothesis of the addition property of equality.

Q6. Solve: $5x + 11 = -24$

Q7. Draw a number-line graph: $|p| > 3$

Q8. Find 20% of 600.

Q9. Does (3, 1) satisfy $4x + 5y = 19$?

Q10. Evaluate: $17 - 7 \cdot 4$

Solve the following systems.

1. $\begin{aligned} x - 2y + 3z &= 3 \\ 2x + y + 5z &= 8 \\ 3x - y - 3z &= -22 \end{aligned}$

2. $\begin{aligned} 2x - y - z &= 7 \\ 3x + 5y + z &= -10 \\ 4x - 3y + 2z &= 4 \end{aligned}$

3. $\begin{aligned} 3x + 2y - z &= 10 \\ x + 4y + 2z &= 3 \\ 2x + 3y - 5z &= 23 \end{aligned}$

4. $\begin{aligned} 3x + 4y + 2z &= 6 \\ x + 3y - 5z &= -7 \\ 5x + 7y - 3z &= 3 \end{aligned}$

5. $\begin{aligned} 5x - 4y + 3z &= 15 \\ 6x + 2y + 9z &= 13 \\ 7x + 6y - 6z &= 6 \end{aligned}$

6. $\begin{aligned} 3x - 2y + 5z &= -17 \\ 2x + 4y - 3z &= 29 \\ 5x - 6y - 7z &= 7 \end{aligned}$

7. $\begin{aligned} 5x - 4y - 6z &= 21 \\ -2x + 3y + 4z &= -15 \\ 3x - 7y - 5z &= 15 \end{aligned}$

8. $\begin{aligned} 2x + 2y + 3z &= -1 \\ 3x - 5y - 2z &= 21 \\ 7x + 3y + 5z &= 10 \end{aligned}$

Problems 9 and 10 have some variables "missing." This makes the systems *easier*, if you are clever enough to figure out *why*.

9. $\begin{aligned} 3x + 4y &= 19 \\ 2y + 3z &= 8 \\ 4x - 5z &= 7 \end{aligned}$

10. $\begin{aligned} 2x - 3y &= 5 \\ 4y + 2z &= -6 \\ 5x + 7z &= -15 \end{aligned}$

The equations in Problems 11 and 12 are either *inconsistent* (*no* common solution), or *dependent* (an *infinite* number of common solutions). By solving the two systems, tell which is which.

11. $\begin{aligned} 6x + 9y - 12z &= 14 \\ 2x + 3y - 4z &= -11 \\ x + y + z &= 1 \end{aligned}$

12. $\begin{aligned} 3x + 2y - z &= 4 \\ 5x - 3y + 2z &= 1 \\ 9x - 13y + 8z &= -5 \end{aligned}$

Problems 13 and 14 are systems with *four* variables.

13. $\begin{aligned} 4w + x + 2y - 3z &= -16 \\ -3w + 3x - y + 4z &= 20 \\ -w + 2x + 5y + z &= -4 \\ 5w + 4x + 3y - z &= -10 \end{aligned}$

14. $$\begin{aligned} w - 5x + 2y - z &= -18 \\ 3w + x - 3y + 2z &= 17 \\ 4w - 2x + y - z &= -1 \\ -2w + 3x - y + 4z &= 11 \end{aligned}$$

Problems 15 and 16 involve fractions. With *constants* in the denominators, as in Problem 15, you may simply *multiply* both members by some number that will get rid of the fractions. With *variables* in the denominators, as in Problem 16, you may first solve for $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, recognizing that $\frac{4}{x}$ is the same as $4(\frac{1}{x})$, etc.

15. $$\begin{aligned} \frac{x}{2} + \frac{y}{4} + \frac{z}{3} &= 24 \\ \frac{x}{4} + \frac{y}{3} + \frac{z}{2} &= 29 \\ \frac{x}{3} + \frac{y}{2} + \frac{z}{4} &= 25 \end{aligned}$$

16. $$\begin{aligned} \frac{4}{x} - \frac{2}{y} + \frac{6}{z} &= -5 \\ -\frac{3}{x} - \frac{5}{y} + \frac{4}{z} &= -3 \\ \frac{2}{x} + \frac{7}{y} - \frac{10}{z} &= 2 \end{aligned}$$

Problems 17 and 18 require you to *find* a system of three equations in three variables. Then you must *solve* the system so that you can answer the problem.

17. A Watusi, a Ubangi, and a Pigmy compare the speeds at which each can run. The sum of the speeds of the natives is 30 miles per hour (mph). The Pigmy's speed plus one third of the Watusi's speed is 22 miles per hour more than the Ubangi's speed. Four times the Watusi's speed plus three times the Ubangi's speed minus twice the Pigmy's speed is 12 miles per hour. Find out how fast each native can run. Then tell what is unfortunate about the Ubangi.
18. The road from Tedium to Ennui is uphill for 5 miles, level for 4 miles, then downhill for 6 miles. John Garfinkle walks from Ennui to Tedium in 4 hours; later he walks halfway from Tedium to Ennui and back again in 3 hours and 55 minutes. Still later he walks from Tedium all the way to Ennui in 3 hours and 52 minutes. What are his rates of walking uphill, downhill, and on level ground, if these rates remain constant?

4-7 SOLUTION OF SECOND-ORDER SYSTEMS BY AUGMENTED MATRICES

There is a methodical way to solve a systems of equations by linear combination. The procedure, called "augmented matrices," may seem to be more tedious at first. But it has the advantage that it can be done by computer. In this section you will learn how to solve two-variable systems this way, and practice using the computer so that you will be prepared to solve systems with three or more variables in the next section.

Suppose that you are to solve the system

$$\begin{aligned} 2x + 3y &= 32 \\ 5x + 4y &= 59 \end{aligned}$$

The coefficient of x can be made zero by multiplying the second equation by 2 and adding -5 times the first equation.

$$0x - 7y = -42$$

The $0x$ term has been left in deliberately so that you can better understand what follows. Dividing each member by -7 gives

$$0x + y = 6$$

So y must equal 6.

The system can be written as follows:

$$\left[\begin{array}{cc|c} 2 & 3 & 32 \\ 5 & 4 & 59 \end{array}\right]$$

The whole thing is called an *augmented matrix*. The coefficients and constants are written in exactly the order they appear in the two equations. The matrix is the square array of numbers to the left of the vertical bar. It has the same pattern as the denominator determinant for the system. The matrix has been "augmented" (added to) by attaching the constants to the right of the vertical bar.

Operations can be performed on the rows of this augmented matrix exactly as they were for the system itself. Multiplying the second row by 2 and subtracting 5 times the first row produces

$$\left[\begin{array}{cc|c} 2 & 3 & 32 \\ 5 & 4 & 59 \end{array}\right] \xrightarrow[+(-5)\times 1]{\text{m2}} \left[\begin{array}{cc|c} 2 & 3 & 32 \\ 0 & -7 & -42 \end{array}\right]$$

The arrow indicates that the second row is being changed. The symbol "m2" above the arrow stands for "multiply each number by 2." The "$+(-5) \times 1$" below the line stands for "add -5 times row 1."

The second row can now be divided by -7. Continuing across the page, you would write

$$\left[\begin{array}{cc|c} 2 & 3 & 32 \\ 5 & 4 & 59 \end{array}\right] \xrightarrow[+(-5)\times 1]{\text{m2}} \left[\begin{array}{cc|c} 2 & 3 & 32 \\ 0 & -7 & -42 \end{array}\right] \xrightarrow{\div(-7)} \left[\begin{array}{cc|c} 2 & 3 & 32 \\ 0 & 1 & 6 \end{array}\right]$$

The second row now says $0x + 1y = 6$, which is equivalent to the solution $y = 6$ found above.

To find x, you make the coefficient of y in the *first* row equal to zero. This can be done by multiplying the first row by 1 and adding -3 times the second row. When you run out of room at the side of the page, just write the instructions at the end of the line, and the result on the lines below.

$$\left[\begin{array}{cc|c} 2 & 3 & 32 \\ 5 & 4 & 59 \end{array}\right] \xrightarrow[+(-5)\times 1]{\text{m2}} \left[\begin{array}{cc|c} 2 & 3 & 32 \\ 0 & -7 & -42 \end{array}\right] \xrightarrow{\div(-7)} \left[\begin{array}{cc|c} 2 & 3 & 32 \\ 0 & 1 & 6 \end{array}\right] \xrightarrow[+(-3)\times 2]{\text{m1}}$$

$$\left[\begin{array}{cc|c} 2 & 0 & 14 \\ 0 & 1 & 6 \end{array}\right] \xrightarrow{\div 2} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 6 \end{array}\right]$$

$S = \{(7, 6)\}$

The completed transformation looks as shown above. The final form of the augmented matrix is equivalent to the system

$$1x + 0y = 7$$
$$0x + 1y = 6$$

When there are 1's on the diagonal from the upper left of the matrix to the lower right, and 0's everywhere else, then the solutions appear in the augmented part of the matrix.

In the following exercise you will get practice "diagonalizing" augmented matrices. ("Matrices" is the plural of "matrix.") Remember as you work these problems that although the method may seem more tedious at first, you are learning it to be able to understand what a computer does in the next section.

EXERCISE 4-7

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. 40 is 20% of what number?

Q2. Solve for p: $rs = pv$

Q3. Associate the 3 and the 5: $4 + 3 + 5$

Q4. Find the slope: $5x + 3y = 30$

Q5. Sketch the graph of a linear function with positive y-intercept and negative slope.

Q6. Sketch a trapezoid.

Q7. Find $f(4)$: $f(x) = 7x - 3$

Q8. Solve: $3x + 5 = 17$

Q9. Simplify: $3x + 5 - 17$

Q10. Multiply: $\left(\frac{2}{3}\right)\left(\frac{5}{7}\right)$

For Problems 1 through 10, solve the system by augmented matrices.

1. $3x + 8y = 54$
   $4x + 5y = 38$
2. $4x + 7y = 68$
   $2x + 5y = 46$
3. $4x + 3y = 29$
   $6x + 7y = 41$
4. $9x + 2y = -16$
   $4x + 3y = -5$
5. $5x - 3y = -7$
   $2x + 5y = 22$
6. $7x + 2y = 13$
   $4x - 5y = -11$
7. $4x - y = -21$
   $3x - 7y = -22$
8. $10x - 3y = 46$
   $x - 7y = 18$
9. $-3x + 2y = 6$
   $-5x - 8y = 10$
10. $-2x - 3y = 15$
    $6x - y = 5$

Problems 11 through 14 are systems whose solutions are not integers. Solve by augmented matrices.

11. $5x + 2y = 43$
    $3x - y = 71$
12. $x - 8y = 31$
    $2x + 5y = -17$
13. $83x + 51y = 463$
    $22x + 19y = 291$
14. $38x + 57y = 1066$
    $92x + 29y = 1492$

Problems 15 through 18 are systems whose equations are either dependent or inconsistent. Try to solve by augmented matrices. Then write a conclusion about how augmented matrices tell you when you have this kind of system, and how they allow you to distinguish between dependent equations and inconsistent equations.

15. $10x + 15y = 21$
    $12x + 18y = 35$
16. $30x + 48y = 126$
    $20x + 32y = 84$
17. $24x + 18y = 66$
    $28x + 21y = 77$
18. $25x + 10y = 37$
    $20x + 8y = 51$

For Problems 19 through 26, solve the system by the interactive computer program MATRIX ROWS on the disk accompanying this text, or similar program.

19. Problem 1, above
20. Problem 2, above
21. Problem 7, above
22. Problem 8, above

23. Problem 13, above
24. Problem 14, above
25. Problem 15, above
26. Problem 16, above

4-8 SOLUTION OF HIGHER-ORDER SYSTEMS BY AUGMENTED MATRICES

Once you understand the augmented matrix procedure in Section 4-7, you can apply it to systems with more than three variables, such as

$$\begin{aligned} 3x - 2y + 5z &= -17 \;\ldots\ldots (1) \\ 2x + 4y - 3z &= 29 \;\ldots\ldots (2) \\ 5x - 6y - 7z &= 7 \;\ldots\ldots (3) \end{aligned}$$

The matrix contains the coefficients and constants as it did for systems with two variables.

$$\left[\begin{array}{rrr|r} 3 & -2 & 5 & -17 \\ 2 & 4 & -3 & 29 \\ 5 & -6 & -7 & 7 \end{array}\right]$$

If you "eliminate" a variable by linearly combining two equations, you are really just making that variable's coefficient equal to *zero*. For instance, eliminating x using Equations (1) and (2) gives

$$\begin{array}{lrcl} (1)\text{m}2: & 6x - 4y + 10z &=& -34 \\ (2)\text{m}-3: & \underline{-6x - 12y + 9z} &=& \underline{-87} \\ & 0x - 16y + 19z &=& -121 \end{array}$$

The instructions to the left say, "Equation (1) multiplied by 2," for example. Replacing the second equation with $0x - 16y + 19z = -121$ gives a different, but equivalent, system, and a new matrix.

$$\begin{aligned} 3x - 2y + 5z &= -17 \\ 0x - 16y + 19z &= -121 \\ 5x - 6y - 7z &= 7 \end{aligned} \qquad \left[\begin{array}{rrr|r} 3 & -2 & 5 & -17 \\ 0 & -16 & 19 & -121 \\ 5 & -6 & -7 & 7 \end{array}\right]$$

So eliminating a variable by linear combination corresponds to *putting a zero* in the appropriate place in the matrix. The operations you can do on the rows of a matrix are the same as the operations you do in linear combination of two equations.

By performing these *row operations* in an appropriate order, you can transform the matrix so that there are zeros everywhere except along the *main diagonal,* and ones there. The above matrix would become

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

The top row of the matrix really says,

$$1x + 0y + 0z = 2,$$

from which you can tell instantly that $x = 2$. Similarly, $y = 4$ and $z = -3$.

The transformed matrix above is said to have been *diagonalized.* The example below shows how you can do the diagonalizing with pencil, paper, and a calculator. Once you understand the process, you can do the operations on a computer so that you won't have to write the entire matrix over again at each step.

Objective:

Given a system of three linear equations with three variables, solve the system by writing it as an augmented matrix and performing the row operations necessary to diagonalize the matrix.

EXAMPLE

Diagonalize the augmented matrix and write the solution set:

$$\left[\begin{array}{ccc|c} 3 & -2 & 5 & -17 \\ 2 & 4 & -3 & 29 \\ 5 & -6 & -7 & 7 \end{array}\right]$$

Solution:

The following steps show what you should write. An instruction such as

$$\xrightarrow[+(1)\text{m}2]{(2)\text{m}-3}$$

indicates that row (2) is to be multiplied by -3, and the result is to be added to row (1) multiplied by 2. If you write this sort of instruction, you can do the operations on a calculator, one entry at a time, without having to write down any intermediate steps. The word *pivot* is used for the row that does *not* change as you do the row operations.

$$\left[\begin{array}{ccc|c} 3 & -2 & 5 & -17 \\ 2 & 4 & -3 & 29 \\ 5 & -6 & -7 & 7 \end{array}\right] \begin{array}{c} \\ \xrightarrow[+\textcircled{1}\text{m}2]{\textcircled{2}\text{m}-3} \\ \xrightarrow[+\textcircled{1}\text{m}5]{\textcircled{3}\text{m}-3} \end{array} \left[\begin{array}{ccc|c} 3 & -2 & 5 & -17 \\ 0 & -16 & 19 & -121 \\ 0 & 12 & 46 & -106 \end{array}\right] \begin{array}{c} \\ \\ \xrightarrow[+\textcircled{2}\text{m}1]{\textcircled{3}\text{m}2} \end{array} \left[\begin{array}{ccc|c} 3 & -2 & 5 & -17 \\ 0 & -16 & 19 & -121 \\ 0 & 0 & 111 & -333 \end{array}\right] \begin{array}{c} \\ \\ \xrightarrow{\textcircled{3}d\,111} \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & -2 & 5 & -17 \\ 0 & -16 & 19 & -121 \\ 0 & 0 & 1 & -3 \end{array}\right] \begin{array}{c} \xrightarrow[+\textcircled{3}\text{m}-5]{\textcircled{1}\text{m}1} \\ \xrightarrow[+\textcircled{3}\text{m}-19]{\textcircled{2}\text{m}1} \\ \\ \end{array} \left[\begin{array}{ccc|c} 3 & -2 & 0 & -2 \\ 0 & -16 & 0 & -64 \\ 0 & 0 & 1 & -3 \end{array}\right] \begin{array}{c} \\ \xrightarrow{\textcircled{2}d-16} \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & -2 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow[+②m2]{①m1} \left[\begin{array}{ccc|c} 3 & 0 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow{①d\,3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$S = \{(2, 4, -3)\}$. ■

In the following exercise you will solve one or two systems with pencil and paper to make sure you understand the concepts. If computers are available, you can solve the other systems by simply instructing the computer what row operation to do, and having it do the work and rewrite the matrix.

EXERCISE 4-8

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Expand the determinant: $\begin{vmatrix} 7 & -3 \\ 5 & 8 \end{vmatrix}$

Q2. Find the x-intercept: $3x - 7y = 42$

Q3. If $y = -3x + 51$, then y varies ______ with x. What word goes in the blank?

Q4. Commute the $8x$ and the $3x$: $5 + 8x + 3x$

Q5. Is $\sqrt{-17}$ an irrational number?

Q6. Find $f(-4)$ if $f(x) = |15 - 3x|$.

Q7. Draw a pair of vertical angles.

Q8. Solve for x: $\begin{aligned} 3x + y &= 17 \\ 2x - y &= 6 \end{aligned}$

Q9. Sketch the graph of a linear function with negative slope and positive y-intercept.

Q10. Find 90% of 900.

Problems 1 through 14 are the same as in Exercise 4-6. For Problems 1 and 2, solve the system by pencil, paper, and calculator using augmented matrices.

1. $\begin{aligned} x - 2y + 3z &= 3 \\ 2x + y + 5z &= 8 \\ 3x - y - 3z &= -22 \end{aligned}$

2. $\begin{aligned} 2x - y - z &= 7 \\ 3x + 5y + z &= -10 \\ 4x - 3y + 2z &= 4 \end{aligned}$

For Problems 3 through 17, solve the system by augmented matrices using the program MATRIX ROWS on the accompanying disk (or similar interactive program). Problems 13, 14, and 17 have systems with more than three variables.

3. $\begin{aligned} 3x + 2y - z &= 10 \\ x + 4y + 2z &= 3 \\ 2x + 3y - 5z &= 23 \end{aligned}$

4. $\begin{aligned} 3x + 4y + 2z &= 6 \\ x + 3y - 5z &= -7 \\ 5x + 7y - 3z &= 3 \end{aligned}$

5. $\begin{aligned} 5x - 4y + 3z &= 15 \\ 6x + 2y + 9z &= 13 \\ 7x + 6y - 6z &= 6 \end{aligned}$

6. $\begin{aligned} 3x - 2y + 5z &= -17 \\ 2x + 4y - 3z &= 29 \\ 5x - 6y - 7z &= 7 \end{aligned}$

7. $\begin{aligned} 5x - 4y - 6z &= 21 \\ -2x + 3y + 4z &= -15 \\ 3x - 7y - 5z &= 15 \end{aligned}$

8. $\begin{aligned} 2x + 2y + 3z &= -1 \\ 3x - 5y - 2z &= 21 \\ 7x + 3y + 5z &= 10 \end{aligned}$

9. $\begin{aligned} 3x + 4y &= 19 \\ 2y + 3z &= 8 \\ 4x - 5z &= 7 \end{aligned}$

10. $\begin{aligned} 2x - 3y &= 5 \\ 4y + 2z &= -6 \\ 5x + 7z &= -15 \end{aligned}$

11. $\begin{aligned} 6x + 9y - 12z &= 14 \\ 2x + 3y - 4z &= -11 \\ x + y + z &= 1 \end{aligned}$

12. $\begin{aligned} 3x + 2y - z &= 4 \\ 5x - 3y + 2z &= 1 \\ 9x - 13y + 8z &= -5 \end{aligned}$

13. $\begin{aligned} 4w + x + 2y - 3z &= -16 \\ -3w + 3x - y + 4z &= 20 \\ -w + 2x + 5y + z &= -4 \\ 5w + 4x + 3y - z &= -10 \end{aligned}$

14. $\begin{aligned} w - 5x + 2y - z &= -18 \\ 3w + x - 3y + 2z &= 17 \\ 4w - 2x + y - z &= -1 \\ -2w + 3x - y + 4z &= 11 \end{aligned}$

15. $\begin{aligned} 2x + 4y - 3z &= 7 \\ 7x - 3y + 2z &= 8 \\ 5x - 5y + 7z &= -1 \end{aligned}$

16. $\begin{aligned} 3x - 7y + 2z &= 11 \\ 8x + 2y - 5z &= -3 \\ 5x - 3y - 3z &= 4 \end{aligned}$

17. $\begin{aligned} 3v - 5w + 2x + 4y + z &= 35 \\ 2v + 4w - x - 3y + 6z &= -16 \\ 4v - 2w - 3x + y + 2z &= 18 \\ -5v + w + 4x - y - 3z &= -18 \\ -2v + 5w + 6x - 2y + z &= -19 \end{aligned}$

4-9 HIGHER-ORDER DETERMINANTS

In Section 4-3 you learned how to use determinants to solve a system of two linear equations in two variables. It is possible to solve a system of any number of linear equations in that same number of variables using higher-order determinants. Although the technique becomes unwieldy for more than three variables, it is interesting to see how what you learned about determinants generalizes to higher order systems.

Objective:

Be able to use determinants to solve a system of three (or more) linear equations with three (or more) variables.

You recall that the system

$$\begin{aligned} 3x + 4y &= -7 \\ 2x - 5y &= 8 \end{aligned}$$

has x and y values as follows:

Constants replace x-coefficients. Constants replace y-coefficients.

$$x = \frac{\begin{vmatrix} -7 & 4 \\ 8 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & -5 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} 3 & -7 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & -5 \end{vmatrix}}$$

x and y coefficients

The denominator determinant contains the x- and y-coefficients from the equations. The x-numerator is obtained by replacing the x-coefficients with the constants -7 and 8 from the right members of the equations. The y-numerator is obtained by replacing the y-coefficients with these constants.

The solutions of a system of three or more linear equations in three or more variables can be written the same way. For instance, the system in the example of Section 4-6 has the following value for y:

System:

$$\begin{aligned} 2x + 3y - z &= -1 \\ -x + 5y + 3z &= -10 \\ 3x - y - 6z &= 5 \end{aligned}$$