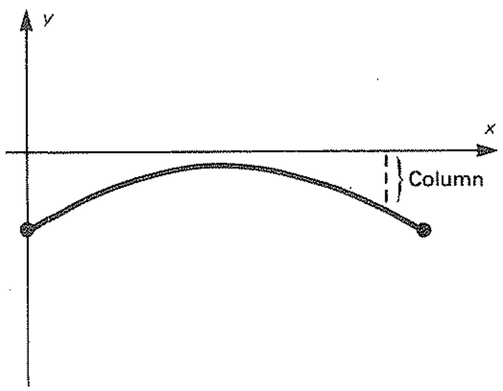
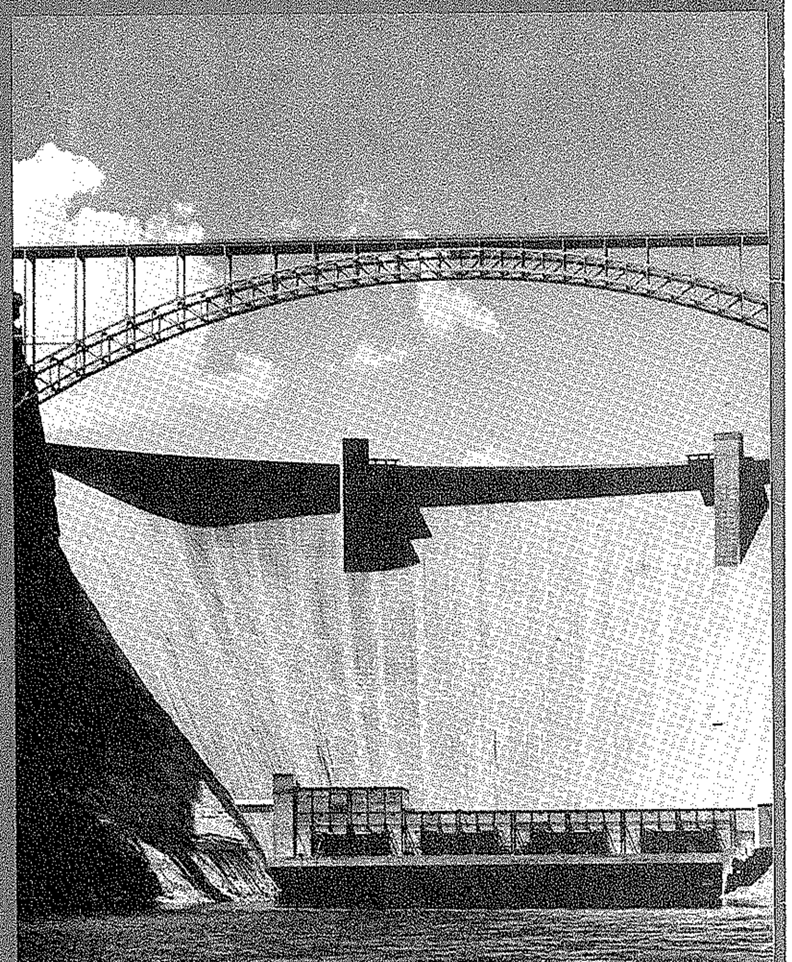
12 to 20 days

5

Quadratic Functions and Complex Numbers

The linear functions in Chapters 3 and 4 had straight-line graphs. In this chapter you will encounter ***quadratic*** *functions whose graphs are* ***curved*** *lines. Along the way you will learn something about imaginary and complex numbers. Again, your ultimate objective will be to find the particular equation of a quadratic function from information about its graph. In Exercise 5-7 you will use such equations in problems ranging from bridge design to predicting the price of pizza!*





5-1 INTRODUCTION TO QUADRATIC FUNCTIONS

You recall from Chapter 3 that a function with an equation such as $y = 8x + 13$ is called a linear function. Functions are named by the kind of expression that y equals. So if

$$y = 3x^2 - 7x + 11,$$

the function is called a quadratic function.

DEFINITION

QUADRATIC FUNCTION
A **quadratic** function is a function whose general equation is

$$\boxed{y = ax^2 + bx + c,}$$

where a, b, and c stand for constants, and $a \neq 0$. (If a were equal to 0, the function would be linear, not quadratic.)

After a new function has been defined, your first task is to explore its graph. In this section you will discover by pointwise plotting what the graph of one quadratic function looks like.

Objective:

Discover by pointwise plotting what the major features of a quadratic function graph are.

The following exercise is designed to help you accomplish this objective.

EXERCISE 5-1

This exercise concerns the graph of $y = x^2 - 6x + 2$.

1. Make a table of values of y for each integer value of x from -2 through 8. Then plot the points on graph paper. If your work is correct, the points should lie along a smooth U-shaped figure called a *parabola*.
2. The low point of the graph is called the *vertex*. Write the coordinates of the vertex as an ordered pair.
3. The graph is symmetrical to a vertical line through the vertex. Draw a dotted line on your graph representing this *axis of symmetry*.
4. What does the y-intercept equal? Where does this number appear in the original equation?
5. There are *two* x-intercepts. What, approximately, do they equal?
6. Why is this function called a *quadratic* function?
7. Explain how the Closure Axioms insure that a quadratic function really is a *function*.

5-2 GRAPHS OF QUADRATIC FUNCTIONS

In Exercise 5-1 you plotted the graph of $y = x^2 - 6x + 2$. The graph is a U-shaped curve called a *parabola*. As shown in Figure 5-2a, this parabola has a low point called the *vertex* at $(3, -7)$. The vertex could also be a high point, as you will soon learn. The vertical line through the vertex is called the *axis of symmetry*. If you fold the graph paper along this line, the two parts will fit on top of each other. In the table of values you can see that the values of y repeat themselves on either side of $x = 3$.

In this section you will learn how to sketch the graph quickly by first calculating the location of the vertex.

Objective:

Given the equation of a quadratic function, calculate the location of the vertex, and use this information to sketch the graph.

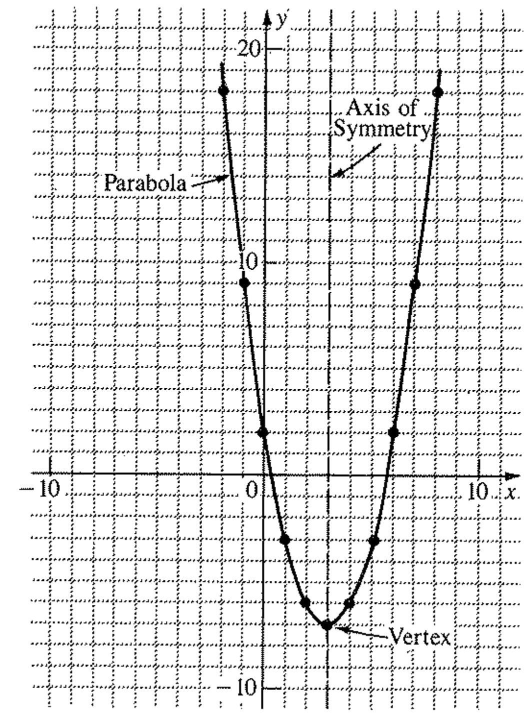


Figure 5-2a

Background: Completing the Square
From previous mathematics courses you should recall how to square a binomial. For example, to square $(x - 5)$, you could write

$(x - 5)^2$	
$= (x - 5)(x - 5)$	Definition of squaring
$= x^2 - 5x - 5x + 25$	Multiply each term of one binomial by each term of the other.
$= x^2 - 10x + 25.$	Combine like terms.

Observe that the "-10" in the answer is twice the "-5" in the original binomial. In general, if the x-coefficient in the binomial you are squaring is equal to 1, then the middle term's coefficient in the answer is *twice* the *constant* term in the binomial.

Suppose that you must find the constant term to add to an expression such as $x^2 + 7x$ in order to make the expression the square of a binomial. Reversing the above observation, the constant in the binomial must be *half* the coefficient of x. So the binomial will be $(x + 3.5)$. The constant to add to $x^2 + 7x$ is thus $(3.5)^2$, or 12.25. So

$$x^2 + 7x + 12.25 = (x + 3.5)^2.$$

The process of adding 12.25 to $x^2 + 7x$ is called *completing the square*.

PROCEDURE

COMPLETING THE SQUARE
If the coefficient of the quadratic term equals 1, as in $x^2 + bx$, then the number that **completes the square** is found by taking *half* of the linear coefficient, b, and *squaring* it. The result is $x^2 + bx + \left(\frac{b}{2}\right)^2$.

To accomplish the above objective, you can transform the given equation by completing the square. For instance, if $y = x^2 - 6x + 2$, you would write

$y = x^2 - 6x + 2$	
$y - 2 = x^2 - 6x$	Subtract 2 from each member to make room to complete the square.
$y - 2 + 9 = x^2 - 6x + 9$	Add 9 to the right member to complete the square. Add 9 to the left member to balance the equation.
$y + 7 = (x - 3)^2$	Simplify and factor.

The coordinates of the vertex, $(3, -7)$, can now be picked out of the equation. The 3 is the value of x that makes the right member of the equation equal zero. The -7 is the value of y that makes the left member equal zero. The equation $y + 7 = (x - 3)^2$ is said to be in *vertex form* because the coordinates of the vertex can be found from it.

CONCLUSION

VERTEX FORM
If the equation of a quadratic function is in the form

$$\boxed{y - k = a(x - h)^2}$$

where a, h, and k are constants, then the vertex is at the point (h, k).

Armed with this conclusion, you can calculate the vertex of any quadratic function graph.

EXAMPLE 1

For $y = -3x^2 - 24x + 11$, transform to vertex form, write the coordinates of the vertex and two other points, and use these to sketch the graph.

Solution:

$y = -3x^2 - 24x + 11$

$y - 11 = -3x^2 - 24x$ Subtract 11 to make room to complete the square.

$y - 11 = -3(x^2 + 8x)$ Factor out -3 so that the x^2-coefficient will equal 1.

$y - 11 + (-3)(16) = -3(x^2 + 8x + 16)$
Add 16 on the right to complete the square. Add $(-3)(16)$ on the left to balance the equation.

$y - 59 = -3(x + 4)^2$ Simplify and factor. This is vertex form.

Vertex: $(-4, 59)$ These values make the right and left members equal zero.

y-intercept $= 11$ Set $x = 0$.

Another point is $(-8, 11)$. This point is directly across the axis of symmetry from the y-intercept.

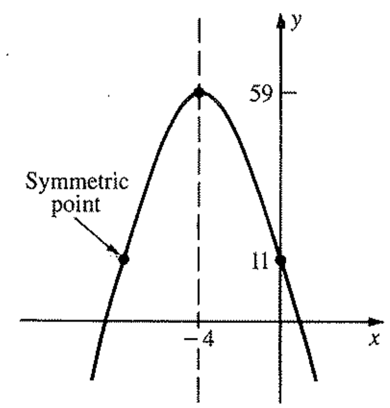


Figure 5-2b

Notes:

1. The parabola in Example 1 opens *downward*. This is what happens if the x^2-coefficient is *negative*.

2. In this section, the point across the axis of symmetry from another point will be called the *symmetric point*.
3. Since only a *sketch* was asked for, you don't need to show scales on the axes. You need show only the coordinates of the vertex, the y-intercept, and the symmetric point.
4. Different scales can be used for the two axes, if necessary, so that the graph will not be too long and skinny. ▨

In the following exercise you will plot several parabolas accurately to be sure you know the shape. Then you will sketch parabolas quickly by first finding the vertex.

EXERCISE 5-2

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Write an equation saying that y is 37% of x.

Q2. Evaluate $7x^2$ if x is -3.

Q3. Simplify: $17 - 7(x - 8)$

Q4. Write the general equation of a linear function.

Q5. Find the area of a triangle with base 10 cm and altitude 9 cm.

Q6. Square 11.

Q7. Sketch the graph of a linear function with negative slope.

Q8. Write $\frac{47}{8}$ as a mixed number.

Q9. State the Associative Axiom for Multiplication.

Q10. Solve: $8x - 13 = 47$

Work the following problems.

For Problems 1 through 4, do the squaring.

1. $(x - 8)^2$
2. $(x + 7)^2$
3. $(5x + 6)^2$
4. $(3x - 9)^2$

For Problems 5 through 8, tell what number must be added to complete the square.

5. $x^2 + 20x +$ ____

6. $x^2 - 24x +$ ____

7. $x^2 - 13x +$ ____

8. $x^2 + 11x +$ ____

For Problems 9 through 12, plot the graph accurately by making a table of values of x and y. You may use different scales on the two axes, if necessary, to make the graphs have reasonable proportions.

9. $y = x^2 - 3x + 5$

10. $y = -x^2 + 5x + 11$

11. $y = -3x^2 + 5x - 10$

12. $y = 5x^2 + 12x - 13$

For Problems 13 through 18, sketch the graph of the quadratic function with the given vertex and intercept.

13. Vertex: $(2, 3)$, y-intercept: 7

14. Vertex: $(-3, 2)$, y-intercept: -4

15. Vertex: $(3, -1)$, y-intercept: -6

16. Vertex: $(-3, -6)$, y-intercept: -4

17. Vertex: $(-3, -4)$, x-intercept: 2

18. Vertex: $(3, 4)$, x-intercept: 1

For Problems 19 through 30, find the vertex, the y-intercept, and symmetric point, and use these to sketch the graph.

19. $y = x^2 + 6x + 11$

20. $y = x^2 + 8x + 21$

21. $y = 3x^2 - 24x + 17$

22. $y = 5x^2 - 30x + 31$

23. $y = 4x^2 + 12x - 11$

24. $y = 8x^2 + 40x + 37$

25. $y = 2x^2 - 11x - 12$

26. $y = 2x^2 - 7x + 12$

27. $y = -5x^2 - 30x + 51$

28. $y = -3x^2 - 24x - 41$

29. $y = -x^2 + x + 1$

30. $y = -x^2 - x + 3$

31. ***Parabola Proportions Problem*** Use the computer program PLOT QUADRATIC on the disk accompanying this text, or a similar function plotter, to plot the following graphs and answer the questions.
 a. Plot $y = x^2$. Which direction does the graph open, up or down?
 b. Plot $y = 0.1x^2$. Does decreasing the x^2-coefficient from 1 to 0.1 make the graph open wider or narrower?
 c. Plot $y = -0.1x^2$. What seems to be true about the graph if the x^2-coefficient is negative? Does the graph seem to have the same proportions as that of $y = 0.1x^2$ from part (b)?

d. Plot $y = 0.1x^2 - 3$. What effect does subtracting 3 have on the proportions and the placement of the graph?
e. Plot $y = 0.1x^2 + 0.6x - 3$. Does adding an x-term affect the proportions of the graph, or just its location on the xy-plane?
f. Suppose that the graph of $y = -0.3x^2 + 2x - 4$ is to be plotted. Make the following predictions about what the graph will look like:
   i. Will it open upward or downward? How do you tell?
   ii. Will it open wide or narrow? How do you tell?
   iii. Will the vertex be on the y-axis or somewhere else? How do you tell?
   iv. Will the graph cross the y-axis above the origin or below? How do you tell?
g. After you have made the predictions in part (f), plot the graph to confirm (or refute!) them.

32. *Squaring Binomials Problem* The following sequence of steps shows why you can square a binomial the short way. Copy the steps on your paper. For each step, name the definition or property that justifies the step.

$(a + b)^2$
a. $= (a + b)(a + b)$
b. $= (a + b)(a) + (a + b)(b)$
c. $= a^2 + ba + ab + b^2$
d. $= a^2 + 2ab + b^2$

33. *General Vertex Form Problem*
a. Transform the general equation $y = ax^2 + bx + c$ to vertex form.
b. Write a formula for h, the x-coordinate of the vertex.
c. Write a formula for k, the y-coordinate of the vertex.
d. Show that the constant a in vertex form is the same number as the coefficient of x^2 in $y = ax^2 + bx + c$ form.

5-3 X-INTERCEPTS, AND THE QUADRATIC FORMULA

Finding the y-intercept of a quadratic function such as $y = 3x^2 + 13x + 7$ is easy. You just substitute 0 for x and get $y = 7$. Finding x-intercepts is harder. Substituting 0 for y gives a quadratic equation such as

$$0 = 3x^2 + 13x + 7.$$

Solving this kind of equation is most easily done using the quadratic formula, which you should have encountered in earlier mathematics courses.

THE QUADRATIC FORMULA
If a quadratic equation has the form $ax^2 + bx + c = 0$, then the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The formula is pronounced, "x equals the opposite of b, plus or minus the square root of the quantity b^2 minus $4ac$, all divided by $2a$."

In case you need a refresher, the formula is derived at the end of this section. The technique used there is completing the square.

For the equation $0 = 3x^2 + 13x + 7$, above, you would simply substitute 3 for a, 13 for b, and 7 for c in the formula, getting

$$x = \frac{-13 \pm \sqrt{169 - 4(3)(7)}}{2(3)}$$

$$x = \frac{-13 \pm \sqrt{85}}{6}$$

$$x = -0.63007\ldots \quad \text{or} \quad -3.70325\ldots$$

In doing the calculation you should be sure to press the [=] key before you divide by 6. Otherwise, the calculator will just divide $\sqrt{85}$ by 6. You know, of course, that there are two separate calculations to do, and that the "±" in the quadratic formula is not the same as the "sign change" key, marked (+/−), on your calculator.

Objective:

Given a quadratic equation, solve it using the quadratic formula, and use the results to find the x-intercepts of a quadratic function.

EXAMPLE 1

Solve $6x^2 - 11x - 5 = 0$.

Solution:
This problem involves a straightforward application of the quadratic formula. Here, $a = 6$, $b = -11$, and $c = -5$.

$$6x^2 - 11x - 5 = 0$$

$$x = \frac{11 \pm \sqrt{121 - 4(6)(-5)}}{2(6)}$$

$$x = \frac{11 \pm \sqrt{241}}{12}$$

$$x = 2.2103\ldots \quad \text{or} \quad x = -0.3770\ldots$$

$$\therefore S = \{2.2103\ldots, -0.3770\ldots\}$$

EXAMPLE 2

Solve $x^2 - 3x + 15 = 0$.

Solution:
The technique again is to apply the quadratic formula, this time with $a = 1$, $b = -3$, and $c = 15$. But there is a surprise!

$$x^2 - 3x + 15 = 0$$

$$x = \frac{3 \pm \sqrt{9 - 4(1)(15)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{-51}}{2}$$

Since $\sqrt{-51}$ is not a real number, there are *no real solutions*. The solutions involve *imaginary* numbers, which, as you recall, are square roots of negative numbers. In the next section you will study these numbers in more detail.

The quantity $b^2 - 4ac$ that appears under the radical sign in the quadratic formula is called the *discriminant*. The name is used because this number "discriminates" between quadratic equations that have real solutions, and those that do not. If all you want to know about a quadratic equation is whether or not it has real solutions, just evaluate the discriminant. If it is negative, there are no real solutions!

DEFINITION

THE DISCRIMINANT
If $ax^2 + bx + c = 0$, then the quantity $b^2 - 4ac$ is called the **discriminant.**

The following are conclusions by which you can tell the kind of numbers the solutions of a quadratic equation will be based on the value of the discriminant.

CONCLUSION

NATURE OF THE SOLUTIONS OF A QUADRATIC EQUATION

Given: $ax^2 + bx + c = 0$, where a, b, and c are real numbers.

If $b^2 - 4ac$ is negative, the equation has solutions with *imaginary* numbers.
If $b^2 - 4ac$ is positive, the equation has *real-number* solutions.
If $b^2 - 4ac$ is a perfect square, and a, b and c are rational numbers, then the solutions are rational numbers.

The second part of the objective is using the quadratic formula to find x-intercepts.

EXAMPLE 3

For $y = x^2 - 5x + 3$, find the vertex, the x-intercepts, and the y-intercept and its symmetric point. Use this information to sketch the graph.

Solution:
Transforming to vertex form gives

$$y + 3.25 = (x - 2.5)^2.$$

So the vertex is at $(2.5, -3.25)$.

Substituting 0 for y in the original equation gives

$$0 = x^2 - 5x + 3.$$

By the quadratic formula,

$$x = \frac{5 \pm \sqrt{25 - 4(1)(3)}}{2(1)}$$

$$x = \frac{5 \pm \sqrt{13}}{2}$$

$$x = 4.3027\ldots \quad \text{or} \quad 0.6972\ldots$$

The y-intercept is 3, and the symmetric point is $(5, 3)$. The graph is shown in Figure 5-3a. ■

There is a way to locate the vertex more quickly than transforming to vertex form. In Figure 5-3a you should be able to notice that the axis of symmetry is halfway between the two x-intercepts. Thus, the x-coordinate of the vertex is the *average* of the x-intercepts.

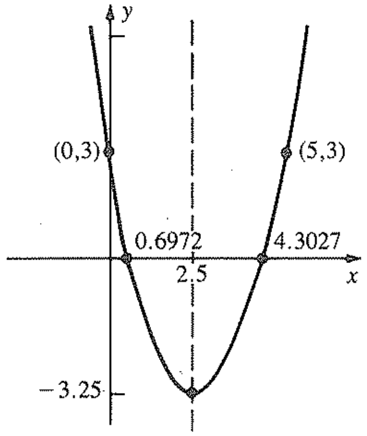


Figure 5-3a

If $y = ax^2 + bx + c$, the x-intercepts are the two values given by the quadratic formula

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If you add these two values of x, the radicals add up to zero. All that is left is $\frac{-b}{2a} - \frac{b}{2a}$, which equals $\frac{-b}{a}$. Dividing by 2 to get the average produces $x = -\frac{b}{2a}$. This number is h, the x-coordinate of the vertex.

CONCLUSION

VERTEX OF A PARABOLA
If $y = ax^2 + bx + c$, then the x-coordinate of the vertex is

$$h = -\frac{b}{2a}$$

EXAMPLE 4

Find the vertex of $y = 7x^2 - 29x + 37$ using the short cut.

Solution:

$$h = -\frac{b}{2a} = -\frac{-29}{2(7)} = 2.071428\ldots$$

Store this value of x in memory. Then substitute it into the original equation to find the y-coordinate, k.

$$k = 7(2.07\ldots)^2 - 29(2.07\ldots) + 37 = 6.964285\ldots$$

$\therefore$ vertex is at $(2.07\ldots, 6.96\ldots)$ ■

In the following exercise you will solve quadratic equations, transform quadratic function equations to vertex form, find the vertex and intercepts, and sketch the graph.

EXERCISE 5-3

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Evaluate $\sqrt{49}$.

Q2. Factor: $x^2 - 10x + 25$

Q3. Multiply: $(2x - 7)(5x + 3)$

Q4. Name by degree and number of terms: $xy^2 + z$

Q5. Find the perimeter of a triangle with sides 3, 4, and 5 inches.

Q6. Find 20% of 600.

Q7. Sketch the graph of a pair of inconsistent linear equations.

Q8. Multiply and simplify: $\left(\frac{6}{7}\right)\left(\frac{3}{14}\right)$

Q9. What property says that x stands for the same number anywhere it appears in an expression?

Q10. Solve: $8x - 10 = 3x$

Work the following problems.

For Problems 1 through 20, solve the equation.

1. $x^2 - 3x + 2 = 0$
2. $x^2 - 5x + 6 = 0$
3. $x^2 + 7x + 12 = 0$
4. $x^2 + 8x + 15 = 0$
5. $x^2 - 2x - 8 = 0$
6. $x^2 - x - 12 = 0$
7. $x^2 + 4x - 3 = 0$
8. $x^2 - 6x + 4 = 0$

9. $2x^2 + 9x - 5 = 0$
10. $3x^2 + 7x + 2 = 0$
11. $3x^2 - 7x = 0$
12. $2x^2 - 15x = 0$
13. $4x^2 - 12x + 9 = 0$
14. $9x^2 + 30x + 25 = 0$
15. $x^2 + 2x + 13 = 0$
16. $x^2 - 10x + 26 = 0$
17. $2x^2 + 2x - 2 = x - x^2$
18. $4x^2 - 8x + 5 = x + 3$
19. $x^2 - 2x + 2 = 2x$
20. $2x^2 - 2x - 2 = x^2$

For Problems 21 through 30, find the discriminant. Then, without actually solving the equation, tell what kind of numbers the solutions will be, real or imaginary. If the solutions are real numbers, tell whether they will be rational or irrational.

21. $3x^2 - 5x + 6 = 0$
22. $5x^2 + 7x - 3 = 0$
23. $2x^2 - 13x + 15 = 0$
24. $9x^2 + 6x + 1 = 0$
25. $10x^2 + 19x + 7 = 0$
26. $x^2 - 6x + 3 = 0$
27. $-3x^2 + 5x - 2 = 0$
28. $x^2 + x + 1 = 0$
29. $-x^2 + 4x - 4 = 0$
30. $x^2 + 6x + 10 = 0$

For Problems 31 through 44, find the vertex, the x- and y-intercepts, and the symmetric point, and sketch the graph.

31. $y = x^2 - 6x + 8$
32. $y = x^2 + 4x + 3$
33. $y = x^2 - 2x - 15$
34. $y = x^2 + 2x - 8$
35. $y = -x^2 - 2x + 3$
36. $y = -x^2 + 4x + 5$
37. $y = 2x^2 + 7x + 3$
38. $y = 3x^2 - 7x + 2$
39. $y = -4x^2 + 4x - 1$
40. $y = x^2 + 6x + 9$
41. $y = x^2 + 2x + 5$
42. $y = -2x^2 + 4x - 3$
43. $y = x^2 + 2x - 5$
44. $y = -x^2 + 4x - 1$

45. ***Solving Quadratics by Completing the Square*** If you did not know the quadratic formula, you could still solve a quadratic equation. All you have to know is how to complete the square. The following example shows you the way.

$5x^2 + 30x + 7 = 0$	The equation to be solved
$5x^2 + 30x = -7$	Clear off space to complete the square.
$x^2 + 6x = -1.4$	Divide by 5 to make the x^2-coefficient equal 1.

$x^2 + 6x + 9 = -1.4 + 9$	(Half of 6)^2 is 9.
$(x + 3)^2 = 7.6$	Factor the left member. Simplify the right member.
$x + 3 = \pm\sqrt{7.6}$	Take the square root of each member.
$x = -3 \pm \sqrt{7.6}$	Subtract 3.
$x = -0.243\ldots$ or $-5.756\ldots$	Arithmetic

Solve the following equations by completing the square.

a. $x^2 + 6x + 4 = 0$
b. $x^2 - 10x + 21 = 0$
c. $7x^2 + 14x + 3 = 0$
d. $x^2 - 7x - 4 = 0$
e. $2x^2 - 10x + 11 = 0$

46. ***Derivation of the Quadratic Formula, Part I*** Solve

$$7x^2 + 13x + 5 = 0$$

by completing the square. Do *not* simplify any of the numbers along the way! For instance, when you divide each member by 7, leave the answer as

$$x^2 + \frac{13}{7}x = -\frac{5}{7}.$$

Your objective is to have the 7, 13, and 5 show up in the answer.

47. ***Derivation of the Quadratic Formula, Part II*** Let a, b, and c be constants, with $a \neq 0$. Solve the equation

$$ax^2 + bx + c = 0$$

by completing the square. Show that the result is equivalent to the quadratic formula.

5-4 IMAGINARY AND COMPLEX NUMBERS

If you solve a quadratic equation such as

$$x^2 - 10x + 34 = 0$$

using the quadratic formula, a negative number appears under the radical sign.

$$x = \frac{10 \pm \sqrt{100 - 4(1)(34)}}{2}$$