$$x = \frac{10 \pm \sqrt{-36}}{2}$$

The symbol $\sqrt{-36}$ means, "a number which, when squared, gives -36 for the answer." Since the square of any real number is *non*-negative, $\sqrt{-36}$ is not a real number. Instead of just giving up, and saying, "The equation has no solutions," mathematicians choose to invent a new kind of number. As you recall from Chapter 1, square roots of negative numbers are called *imaginary numbers*.

To make sense out of imaginary numbers, the first step is to define a number whose square root is -1. This number is called i (for "imaginary").

DEFINITION

UNIT IMAGINARY NUMBER
i is a number whose square is -1. That is,

$$\boxed{i^2 = -1}$$

Notes:

1. Since $i^2 = -1$, it is customary to write

$$\boxed{i = \sqrt{-1}}$$

2. The reason for the name "imaginary" number is that when they were proposed several hundred years ago, people could not "imagine" such a number. However, they are no less real than "real" numbers since both kinds of number are inventions of *people*.
3. The number i is called the *unit* imaginary number, just as 1 (or $\sqrt{+1}$) is called the unit *real* number.

A number such as $\sqrt{-36}$ can now be defined in terms of i. It can be written this way:

$$\begin{aligned} &\sqrt{-36} \\ &= \sqrt{(-1)(36)} \\ &= \sqrt{-1}\,\sqrt{36} \\ &= i\sqrt{36}, \quad \text{or more simply, } 6i. \end{aligned}$$

From this example, the definition follows.

DEFINITION

IMAGINARY NUMBERS IN TERMS OF i
If x is a non-negative real number, then

$$\sqrt{-x} = i\sqrt{x}$$

So any imaginary number is the product of a *real* number and the unit imaginary number i.

With the aid of the above definition, you can now write the solution to the above equation in terms of i.

$$x = \frac{10 \pm \sqrt{-36}}{2}$$

$$x = \frac{10 \pm 6i}{2}$$

$$x = 5 \pm 3i$$

The sum of a real number and an imaginary number, such as $5 + 3i$, is called a *complex* number. The 5 is called the *real* part of $5 + 3i$ and the coefficient 3 is called the *imaginary part* of $5 + 3i$.

DEFINITION

COMPLEX NUMBERS
A **complex number** is a number of the form $a + bi$, where the real number a is called the *real part* of $a + bi$, the real number b is called the *imaginary part* of $a + bi$, and i is $\sqrt{-1}$.

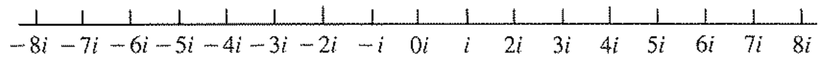
The two numbers $5 + 3i$ and $5 - 3i$, which are the solutions of the equation at the beginning of this section, are called *complex conjugates* of each other. If a quadratic equation has real numbers for its coefficients, then its solutions will always be complex conjugates.

DEFINITION

COMPLEX CONJUGATES
The complex numbers $a + bi$ and $a - bi$ are called **complex conjugates** of each other.

The next thing to do is see how complex numbers fit in with the real numbers you have always used. Imaginary numbers can be plotted on their own number line. The scale is marked off in multiples of i, like this:

Imaginary Number Line



If you consider $0i$ and 0 to be the same number, the imaginary-number line and real-number line can be crossed at their origins. The result is a Cartesian coordinate system, shown in Figure 5-4a.

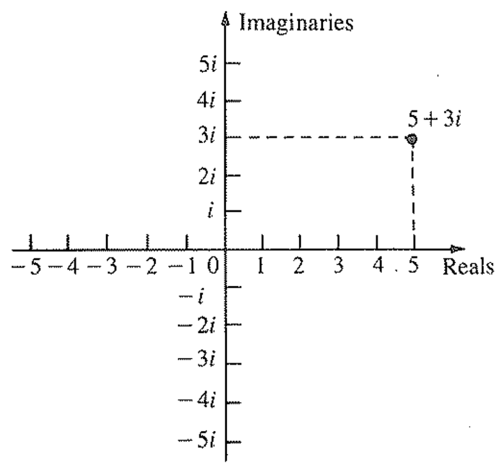


Figure 5-4a

To graph a complex number like $5 + 3i$, you go across 5 in the "real" direction, then up 3 in the "imaginary" direction. This number is plotted in Figure 5-4a.

The coordinate system is called the *complex-number plane*. If you will back off and look at it from a distance, you should be able to see that the real numbers are just a subset of the complex numbers. The imaginary and complex numbers fill in the space above and below the old, familiar number line. All along you have been operating in the set of complex numbers, but perhaps had never thought to look *above* and *below* the real-number line to see what was there!

Objective:

Given a quadratic equation whose solutions are complex numbers, write the solutions in terms of i, and check them by substitution.

EXAMPLE 1

Solve $x^2 - 10x + 34 = 0$ and check one of the solutions.

Solution:
From the work above, the solutions are $5 + 3i$ and $5 - 3i$.

$\therefore S = (5 + 3i, 5 - 3i)$

Check of $5 - 3i$:

$(5 - 3i)^2 - 10(5 - 3i) + 34$

$= 25 - 30i + 9i^2 - 50 + 30i + 34$ Square the binomial and distribute the -10.

$= 9 + 9i^2$ Combine like terms.

$= 9 - 9$ Because $i^2 = -1$

$= 0$, which checks. ■

The main thing for you to remember in doing operations with complex numbers is that i^2 is defined to be -1. Otherwise, all of the properties and axioms apply as for real numbers.

EXAMPLE 2

Solve $7x^2 + 8x + 25 = 5x + 6$, and check one solution.

Solution:
Before you can use the quadratic formula, one side of the equation must equal zero.

$7x^2 + 8x + 25 = 5x + 6$ Write the given equation.

$7x^2 + 3x + 19 = 0$ Subtract $5x$ and subtract 6.

$x = \dfrac{-3 \pm \sqrt{9 - 4(7)(19)}}{14}$ Use the quadratic formula.

$x = \dfrac{-3 \pm \sqrt{-523}}{14}$ Simplify the radical.

$x = \dfrac{-3 \pm i\sqrt{523}}{14}$ Definition of imaginary numbers

$x = -0.2142... \pm 1.6335...i$ By calculator

$\therefore S = (-0.2142... + 1.6335...i, -0.2142... - 1.6335...i)$

Check: (Use $0.2142... \pm 1.6335...i$.)

If your calculator has a complex number mode, you can do the check directly. If not, and your calculator has two memories, store the real part in one memory and the imaginary part in the other. Remember, $i^2 = -1$.

$$7(-0.2142\ldots + 1.6635\ldots i)^2 + 8(-0.2142\ldots + 1.6635\ldots i) + 25 \stackrel{?}{=} 5(-0.2142\ldots + 1.6635\ldots i) + 6$$

$$7(0.0459\ldots - 0.70007\ldots i - 2.6683\ldots) - 1.7142\ldots - 13.0681\ldots i + 25 \stackrel{?}{=} -1.0714\ldots + 8.1675\ldots i + 6$$

$$0.3214\ldots - 4.90054\ldots i - 18.6785\ldots - 1.7142\ldots - 13.0681\ldots i + 25 \stackrel{?}{=} 4.9285\ldots + 8.1675\ldots i$$

$4.9285\ldots + 8.1675\ldots i = 4.9285\ldots + 8.1675\ldots i$, which checks. ■

In the following exercise you will solve and check some quadratic equations, and gain a bit more insight into the nature of complex numbers.

EXERCISE 5-4

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Do the squaring: $(p + 5)^2$

Q2. Do the squaring: $(y - 7)^2$

Q3. Do the squaring: $(3x + 8)^2$

Q4. Do the squaring: 13^2

Q5. Sketch the graph of a quadratic function opening downward.

Q6. Sketch the graph of a linear function with negative slope.

Q7. Find the discriminant of $x^2 + 7x + 20 = 0$.

Q8. Evaluate the determinant: $\begin{vmatrix} 3 & 7 \\ 4 & 9 \end{vmatrix}$

Q9. Find 2% of 700.

Q10. Add: $\frac{2}{3} + \frac{3}{4}$

For Problems 1 through 12,
a. Solve the equation.
b. Check one of the solutions.

1. $x^2 - 14x + 58 = 0$

2. $x^2 - 6x + 73 = 0$

3. $x^2 - 10x + 26 = 0$
4. $x^2 - 14x + 50 = 0$
5. $9x^2 + 12x + 68 = 0$
6. $9x^2 + 90x + 226 = 0$
7. $2x^2 - 3x - 5 = 0$
8. $4x^2 - 21x - 18 = 0$
9. $x^2 - 3x + 41 = x + 12$
10. $x^2 + 5x + 50 = 3x - 15$
11. $3x(x + 5) + 2x^2 = 8x - 11$
12. $8(x - 1)^2 = 7x - 32$

For Problems 13 through 20, plot the complex number on a complex-number plane.

13. $4 + 9i$
14. $6 + 2i$
15. $-3 + 5i$
16. $5 - 7i$
17. $7 - 10i$
18. $-6 + i$
19. $-1 - 2.6i$
20. $-3.2 - 4i$

21. ***Quadratic Function Intercepts Problem*** The following quadratic functions differ only in the constant term.

$$f(x) = x^2 - 6x + 5$$
$$g(x) = x^2 - 6x + 9$$
$$h(x) = x^2 - 6x + 13$$

a. Find the x-intercepts of each function.
b. Draw the graph of each function. You may make a sketch using information you have already found, or use a computer graphics program such as PLOT QUADRATIC on the accompanying disk.
c. What is true about the graph of a quadratic function if the x-intercepts are both *real* numbers? both non-real *complex* numbers? both *equal* to each other?

22. Why do you suppose part (c) of Problem 21, above, says *non-real* complex numbers, not just complex numbers?

23. ***Complex Conjugates Problem***
a. Write the complex conjugate of $4 + 7i$.
b. Write the complex conjugate of $3 - 8i$.
c. Do the multiplying: $(7 + 3i)(7 - 3i)$. What do you notice about the answer?
d. Do the addition: $(11 + 5i) + (11 - 5i)$. What do you notice about the answer?
e. Do the subtraction: $(6 + 10i) - (6 - 10i)$. What do you notice about the answer?
f. Prove that the sum and the product of two complex conjugates is always a *real* number, and the difference between a complex number and its conjugate is always a pure imaginary number.

24. ***Complex Conjugates and Quadratic Equations Problem*** Prove that if $ax^2 + bx + c = 0$, where a, b, and c are real numbers, and the discriminant is negative, then the two solutions are complex conjugates of each other.

25. ***Powers of* i *Problem*** The definition of i makes i^2 equal to -1. Since $i^3 = i^2 \cdot i$, it follows that $i^3 = -i$.
    a. Evaluate each positive integer power of i from i^4 through i^{10}.
    b. Describe the pattern that shows up in the powers of i.
    c. Show that i and i^0 both fit the pattern in part (a).
    d. Quick! Tell what i^{100} will equal.
    e. What will i^{2001} equal? What will i^{137} equal? What will i^{50} equal?

5-5 EVALUATING QUADRATIC FUNCTIONS

When you use a function as a mathematical model, you must be able to calculate y when x is known, and be able to calculate x when y is known. In this section you will practice these things so that you will be comfortable doing them in the problems of the next section.

Objective:

Given the equation of a quadratic function, be able to calculate the value of y for a known value of x, and the value(s) of x for a known value of y.

EXAMPLE 1

If $f(x) = 3x^2 + 2x - 11$, find
a. $f(-4)$
b. x, if $f(x) = -6$, and
c. the x-intercepts.

Solution:

a. If you wish, you may do the whole calculation on your calculator without writing down intermediate results. Otherwise, the steps are:

$$\begin{aligned} f(-4) &= 3(-4)^2 + 2(-4) - 11 \\ &= 3(16) + 2(-4) - 11 \\ &= 48 - 8 - 11 \\ &= \underline{\underline{29}} \end{aligned}$$

b. Substituting -6 for $f(x)$ leads to a quadratic equation. Before you use the quadratic formula you must transform the equation so that one member equals zero.

$$-6 = 3x^2 + 2x - 11$$
$$3x^2 + 2x - 5 = 0$$
$$x = \frac{-2 \pm \sqrt{4 - 4(3)(-5)}}{2(3)}$$
$$x = \frac{-2 \pm \sqrt{64}}{6}$$
$$x = 1 \quad \text{or} \quad x = \frac{-5}{3}$$

Note that since the equation said, "Find the values of x. . . ," not "Solve the equation. . . ," it is not necessary to write the answer as a solution set.

c. As you recall, an x-intercept is a value of x when $y = 0$. Setting $f(x) = 0$ gives

$$0 = 3x^2 + 2x - 11$$
$$x = \frac{-2 \pm \sqrt{4 - 4(3)(-11)}}{2(3)}$$
$$x = \frac{-2 \pm \sqrt{136}}{6}$$
$$x = 1.6103\ldots \quad \text{or} \quad x = -2.2769\ldots$$

Sometimes all you are interested in knowing about a function is whether or not y ever equals a particular given value. As shown in Figure 5-5a, the parabola might not get high enough or low enough to reach the given y-value. One way to find out is to set y equal to the given value and solve

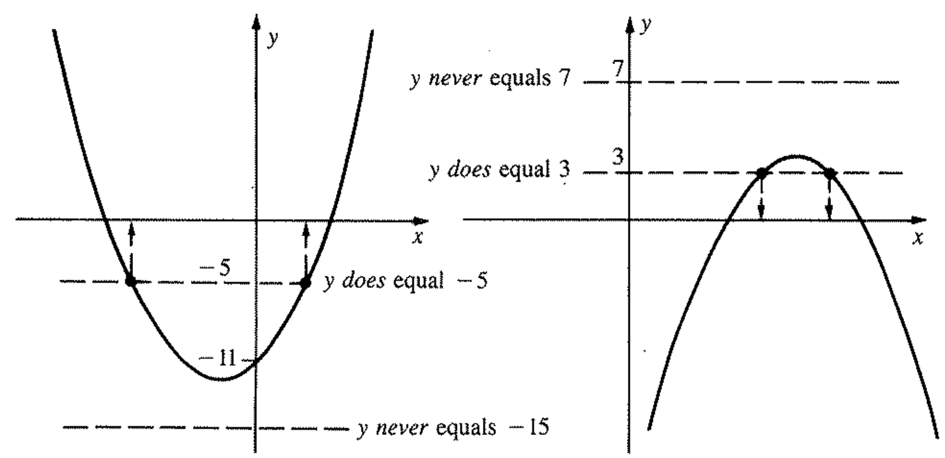


Figure 5-5a

the equation for x. If there are *real* values of x, then y *does* reach the given value. If the values of x are non-real complex numbers, then y does *not* reach the given value.

EXAMPLE 2

If $f(x) = 3x^2 + 2x - 11$, does $f(x)$ ever equal:

a. -5?

b. -15?

Solution:

a. Setting $f(x) = -5$ gives:

$-5 = 3x^2 + 2x - 11$

$3x^2 + 2x - 6 = 0$

$b^2 - 4ac = 4 - 4(3)(-6)$ Definition of discriminant

$= 76.$

Since the discriminant is positive, there are *real* values of x for which $f(x) = -5$.

b. Setting $f(x) = -15$ gives:

$-15 = 3x^2 + 2x - 11$

$3x^2 + 2x + 4 = 0$

$b^2 - 4ac = 4 - 4(3)(4)$

$= -44.$

Since the discriminant is negative, the solutions of the equation will be (non-real) complex numbers. So $f(x)$ never reaches -15. ■

EXERCISE 5-5

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Square -3.

Q2. Square $5i$.

Q3. Square $x - 7$.

Q4. Factor: $x^2 + 3x - 40$

Q5. Find the slope: $3x + 7y = 42$

Q6. Find the x-coordinate of the vertex: $f(x) = 5x^2 - 30x + 17.9$

Q7. 30 is 60% of what number?

Q8. Multiply: $\left(\frac{3}{7}\right)\left(\frac{2}{3}\right)$

Q9. Solve for y: $x + y = 5$
$x - y = 2$

Q10. Sketch the graph of a quadratic function with vertex below the x-axis and no real x-intercepts.

Work the following problems.

1. Suppose that $f(x) = 5x^2 + 8x - 7$.
   a. Find $f(-3)$.
   b. Find x when $f(x) = -3$.
   c. Find the x-intercepts.
2. Suppose that $g(x) = 2x^2 - 5x - 11$.
   a. Find $g(-4)$.
   b. Find x when $g(x) = -4$.
   c. Find the x-intercepts.
3. Suppose that $h(x) = -2x^2 + 3x - 10$.
   a. Find $h(-9)$.
   b. Find x when $h(x) = -9$.
   c. Find the x-intercepts.
4. Suppose that $f(x) = -4x^2 + 4x + 15$.
   a. Find $f(-3)$.
   b. Find x when $f(x) = 20$.
   c. Find the x-intercepts.
5. Suppose that $y = x^2 + 8x + 15$. Find the value(s) of x for which
   a. $y = 3$, b. $y = 2$, c. $y = 0$,
   d. $y = -1$, e. $y = -3$, f. $y = 15$.
6. Suppose that $y = -x^2 - 6x + 5$. Find the value(s) of x for which
   a. $y = 5$, b. $y = 4$, c. $y = 3$,
   d. $y = 2$, e. $y = 0$, f. $y = -5$.

For Problems 7 through 14, use the discriminant to tell whether or not the indicated function ever has the given values of y (for *real* values of x).

7. $y = 4x^2 - 7x + 2$; $y = 5$, $y = -3$.
8. $y = 3x^2 + 10x - 1$; $y = 6$, $y = -4$.

9. $y = 2x^2 + 3x + 6;\quad y = 1,\quad y = -5.$
10. $y = 5x^2 - 8x + 6;\quad y = 3,\quad y = -4.$
11. $y = -3x^2 + 5x + 1;\quad y = 4,\quad y = -3.$
12. $y = -2x^2 + 6x - 7;\quad y = 10,\quad y = -10.$
13. $y = -x^2 + 10x - 8;\quad y = 7,\quad y = 0.$
14. $y = -x^2 - 6x - 9;\quad y = 1,\quad y = 0.$
15. ***Graphs of Complex Solutions Problem*** The graph of

$$f(x) = x^2 - 6x + 34$$

opens upward, and has a vertex above the x-axis. So the x-intercepts turn out to be complex numbers. If you stretch out the x-axis into a complex-number plane, you get the three-dimensional graph shown in Figure 5-5b. The parabola is in the plane of the x-axis and $f(x)$-axis. Below the vertex, there is another parabola. It is in a plane perpendicular to the real x-axis. The two intercepts lie at the points where this second parabola pierces the complex x-plane. Answer the following questions.

a. Set $f(x)$ equal to zero, and thus show that the x-intercepts really are $3 + 5i$ and $3 - 5i$.
b. Show that $f(3 + 5i)$ really does equal zero.
c. Show that $f(3 + 2i)$ is a real number.
d. Show that $f(7 + 4i)$ is *not* a real number.
e. Make a conjecture about values of a and b for which $f(a + bi)$ is a real number. Explain how you arrived at your conjecture.
f. Prove that your conjecture in part (e) is true.

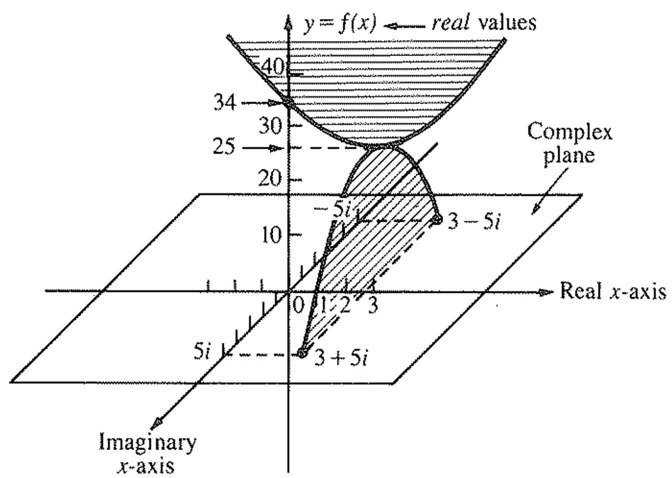


Figure 5-5b

5-6 EQUATIONS OF QUADRATIC FUNCTIONS FROM THEIR GRAPHS

In order to use a quadratic function as a mathematical model of something in the real world, you must be able to find the particular equation from information about the graph. For linear functions, you needed only two ordered pairs. For quadratic functions, it takes three ordered pairs.

Objective:

Given three points on the graph of a quadratic function, or the vertex and one other point, find the particular equation of the function.

EXAMPLE 1

Find the particular equation of the quadratic function containing $(-2, -11)$, $(4, 13)$, and $(6, 29)$.

Solution:
The graphs of the three given points are shown in Figure 5-6a. Since they do not lie in a straight line, there is a quadratic function whose graph contains the three points.

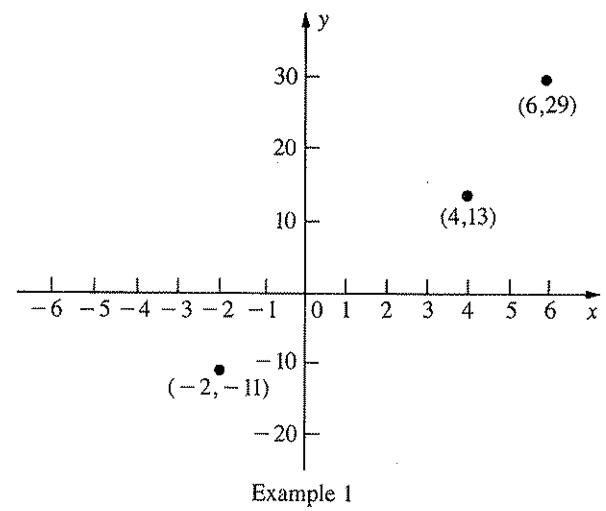


Figure 5-6a

The first thing to do is write the general equation.

$$y = ax^2 + bx + c$$

Substituting the first ordered pair, $(-2, -11)$ for (x, y) gives

$$-11 = a(-2)^2 + b(-2) + c,$$

which can be transformed to

$$4a - 2b + c = -11 \quad \ldots\ldots \quad (1)$$

This is a linear equation in the "variables" a, b, and c. Substituting the other two ordered pairs, $(4, 13)$ and $(6, 29)$, gives two more equations.

$$16a + 4b + c = 13 \quad \ldots\ldots (2)$$

$$36a + 6b + c = 29 \quad \ldots\ldots (3)$$

The system formed by Equations (1), (2), and (3), can be solved by linear combinations.

$$\begin{array}{l} 4a - 2b + c = -11 \\ 16a + 4b + c = 13 \\ 36a + 6b + c = 29 \end{array}$$

$$12a + 6b = 24 \longrightarrow 2a + b = 4$$

$$20a + 2b = 16 \longrightarrow 10a + b = 8$$

$$8a = 4$$

$\therefore a = 0.5$

Substituting 0.5 for a in a two-variable equation gives

$$1 + b = 4$$

$$b = 3.$$

Substituting 0.5 for a and 3 for b in the first equation gives

$$2 - 6 + c = -11$$

$$c = -7$$

So the desired equation is $y = 0.5x^2 + 3x - 7$.
Note that the answer is an *equation*, not a solution set.

EXAMPLE 2

Find the particular equation of the quadratic function containing $(0, 5)$, $(2, 13)$, and $(3, 26)$.

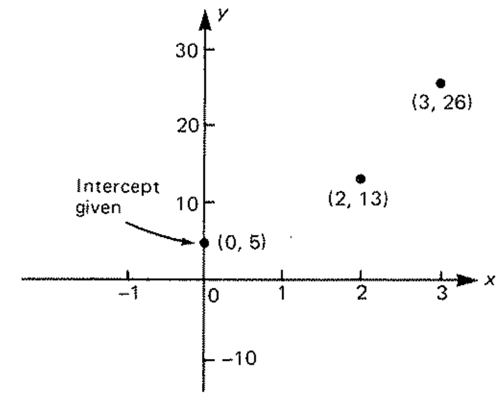
Solution:
The job of finding this equation is easier because you know the y-intercept. Substituting $(0, 5)$ for (x, y) in $y = ax^2 + bx + c$ gives

$$5 = a \cdot 0^2 + b \cdot 0 + c,$$

from which $c = 5$. Using this value of c when you substitute the other two ordered pairs gives *two* equations in a and b:

$13 = 4a + 2b + 5$ Substitute $(x, y) = (2, 13)$ and $c = 5$.

$26 = 9a + 3b + 5$ Substitute $(x, y) = (3, 26)$ and $c = 5$.



Example 2

Figure 5-6b

which can be transformed to

$$2a + b = 4 \quad \text{\_\_\_\_} \quad ①$$

$$3a + b = 7 \quad \text{\_\_\_\_} \quad ②.$$

Multiplying ① by -1 and adding it to ② gives

$$a = 3.$$

Substituting 3 for a in ① gives

$$6 + b = 4,$$

$$b = -2.$$

So the desired equation is

$$\underline{\underline{y = 3x^2 - 2x + 5.}}$$

EXAMPLE 3

Find the particular equation of the quadratic function with vertex at $(2, -5)$ and containing $(3, 1)$.

Solution:
If you know that one of the given points is the *vertex*, then it would be easier to use the *vertex* form, $y - k = a(x - h)^2$. Substituting $(2, -5)$ for the vertex (h, k) gives

$$y - (-5) = a(x - 2)^2.$$

The only constant left to be evaluated is a. This is why you need only *one* other ordered pair. Substituting $(3, 1)$ for (x, y) gives

$$1 - (-5) = a(3 - 2)^2$$

$$6 = a.$$

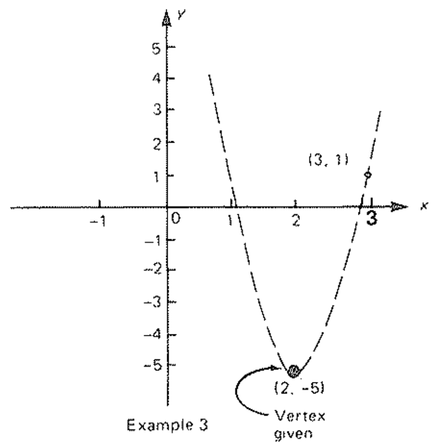


Figure 5-6c

Therefore, the equation is

$$\underline{y + 5 = 6(x - 2)^2}.$$

If desired, this equation can be transformed to

$$\underline{y = 6x^2 - 24x + 19}.$$

The exercise which follows is designed to give you practice in finding the equation of a quadratic function from information about its graph. This is the fundamental technique which you will use in Section 5-8 for making quadratic mathematical models of situations in the real world.

EXERCISE 5-6

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

Q1. Does $7x^2 + 3x + 5 = 0$ have real-number solutions?

Q2. Find the vertex: $y = x^2 - 6x + 2$

Q3. Find the x-intercept: $3x + 4y = 24$

Q4. Sketch the graph of a quadratic function opening downward.

Q5. Find $f(2)$ if $f(x) = x^3$.

Q6. Add: $\frac{4}{9} + \frac{1}{2}$

Q7. Find 20% of 3.

Q8. Is $-\sqrt{81}$ a rational number?

Q9. Sketch the graphs of two inconsistent linear equations.

Q10. What special name is given to the domain in a linear programming problem?

For Problems 1 through 14, find the particular equation of the quadratic function containing the given ordered pairs. Write the equation in the form $y = ax^2 + bx + c$.

1. (1, 6), (3, 26), (−2, 21)
2. (1, 2), (−2, 23), (3, 8)
3. (−2, −41), (−3, −72), (5, −48)
4. (−3, 18), (6, −9), (12, −57)
5. (4, 7.3), (6, 12.7), (−3, 1.0)
6. (10, 1), (20, 22), (−30, −3)
7. (10, 40), (−20, 160), (−5, 10)
8. (2, −2.8), (−3, −6.3), (5, −17.5)
9. (−4, −37), (2, 11), (0, −1)
10. (0, 5), (4, 1), (−3, −13)
11. (0, 0), (−1, 7), (6, 42)
12. (0, 0), (−1, 4), (3, −48)
13. Vertex at (−4, 3), also containing (−6, 11)
14. Vertex at (−2, 3), also containing (4, 12)
15. Show that there is *no* quadratic function which contains the points (5, 2), (6, −4), and (5, −7). Explain what it is about these three points that prevents there being a quadratic function containing all of them.
16. Show that there is *no* quadratic function which contains the points (−2, −1), (1, 8), and (3, 14). Explain what it is about these three points that prevents there being a quadratic function containing all of them.

5-7 QUADRATIC AND LINEAR FUNCTIONS AS MATHEMATICAL MODELS

Now that you know how to find the particular equation of a quadratic function from points on its graph, you can use this kind of function as a mathematical model of the relationship between two real-world variables. Quadratic functions are reasonable models where the graph is curved rather than straight. They are especially appropriate if the graph has a high point or a low point. As you recall from Chapter 3, linear functions are more appropriate where the graph is a straight line.

Objective:

Be able to use a quadratic function or a linear function as a mathematical model for a real-world situation, depending on which one is called for.

EXAMPLE 1

An old Pizza Inn menu from the 1960's lists the following prices for plain cheeze pizzas:

Small (8″ diameter) \$0.85
Medium (10″ diameter) \$1.15
Large (13″ diameter)........ \$1.75

a. Assume that the price is a quadratic function of the diameter. Write the particular equation expressing price in terms of diameter.
b. If Pizza Inn had made 20″ pizzas, what do you predict the price would have been?
c. Suppose that the menu had listed a "Colossal" pizza costing \$6.00. What do you predict its diameter would have been?
d. The price-intercept is the price when the diameter is zero. What does the price-intercept equal in this mathematical model? Why do you suppose that it is greater than zero?
e. Use the discriminant to show that there are no diameters for which the price is zero.
f. Show that a linear function does *not* fit the original data.
g. Find the vertex. Use it and other points that are given or that have been calculated to sketch the graph.

Solutions:

a. Let p = number of *cents* for a pizza.
Let d = number of inches diameter.
General equation: $p = ad^2 + bd + c$
Ordered pairs: (8, 85), (10, 115), (13, 175)

(Note: Once you get to this point, you are out of the real world and into the mathematical world. The rest of the problem involves using mathematical techniques with which you should now be familiar, and interpreting answers you get in the mathematical world apply to the real world.)

Substituting the three ordered pairs gives

$$64a + 8b + c = 85$$
$$100a + 10b + c = 115$$
$$169a + 13b + c = 175$$

Solving this system, as in the previous section, gives

$$a = 1, \quad b = -3, \quad c = 45.$$

Equation is: $p = d^2 - 3d + 45$

b. Substitute 20 for d.

$$p = 20^2 - 3(20) + 45$$
$$= 385$$

20-inch pizza would have cost about $3.85

c. Substitute 600 for p.

$$d^2 - 3d + 45 = 600$$
$$d^2 - 3d - 555 = 0$$
$$d = \frac{3 \pm \sqrt{9 - 4(1)(-555)}}{2(1)}$$
$$= \frac{3 \pm \sqrt{2229}}{2}$$
$$= 25.106\ldots \text{ or } -22.106\ldots$$

Diameter would have been about 25″.

(Note that the negative solution is meaningless this time. The precise answer 25.106... from the mathematical world should be rounded off to something that is appropriate for the real world.)

d. Let $d = 0$.
Price-intercept is 45.
It is greater than zero because there are fixed charges, such as cooking, serving, and washing dishes, which do not depend on the size of the pizza, just on the fact that you ordered a pizza.

e. Let $p = 0$.
$d^2 - 3d + 45 = 0$
Discriminant $= (-3)^2 - 4(1)(45) = -171$
No real solutions. Therefore, there are no diameters for which the price is zero.

f. From (8, 85) to (10, 115), the slope of the line would be $\frac{30}{2} = 15$.

From (10, 115) to (13, 175), the slope would be $\frac{60}{3} = 20$. So a linear function does not fit because the slopes are not equal.

g. The horizontal coordinate of the vertex is

$$d = \frac{-b}{2a} = -\frac{-3}{2} = 1.5.$$

Substituting 1.5 for d gives

$$p = (1.5)^2 - 3(1.5) + 45 = 42.75.$$

Vertex is at (1.5, 42.75).

The graph is shown in Figure 5-7a.

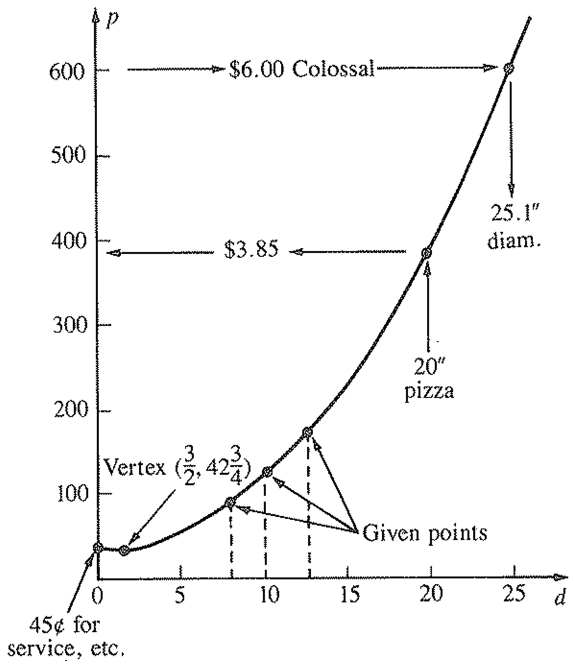


Figure 5-7a

EXAMPLE 2 ***Rectangular Walkway Problem***

A rectangular pond 5 meters by 7 meters is to be surrounded by a walkway of width x meters (see Figure 5-7b). This problem concerns the rectangular region taken up by the pond and walkway.

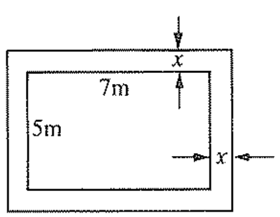


Figure 5-7b

a. Write the length and width of the region as functions of x. What kind of functions are these?
b. Write the area of the region as a function of x. What kind of function is this?
c. Predict the area of the region if $x = 1$, 2, and 3.
d. Find the value of x which makes the region have an area of 150 square meters.
e. Sketch the graph in a reasonable domain.
f. Find the value of x which makes the walkway have an area equal to the area of the pond.

Solutions:

a. x = number of meters wide the strip is.
$7 + 2x$ = number of meters long the rectangle is.
$5 + 2x$ = number of meters wide the rectangle is.
These are *linear functions* of x.

b. Let A = number of square meters in the area of the rectangle.

$$\therefore A = (7 + 2x)(5 + 2x)$$

$$A = 4x^2 + 24x + 35$$

This is a *quadratic function*.

c.

x	Area
1	63
2	99
3	143

d. $150 = 4x^2 + 242x + 35$

$$4x^2 + 24x - 115 = 0$$

$$x = \frac{-24 \pm \sqrt{576 - 4(4)(-115)}}{8}$$

$$x = \frac{-24 \pm \sqrt{2416}}{8}$$

Out of domain

$x = 3.144\ldots$ or ~~$-9.144\ldots$~~

About 3.14 m

e. See Figure 5-7c.

f. If the area of the walkway equals the area of the pond, then the area of the total rectangle is twice the area of the pond, or 70 square meters.

$$70 = 4x^2 + 24x + 35$$

$$4x^2 + 24x - 35 = 0$$

$$x = \frac{-24 \pm \sqrt{576 - 4(4)(-35)}}{8}$$

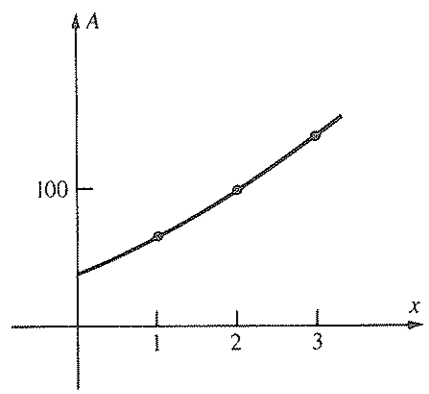


Figure 5-7c

$$x = \frac{-24 \pm \sqrt{1136}}{8}$$

Out of domain

$x = 1.213\ldots$ or ~~$-7.213\ldots$~~

About 1.21 m

In the following exercise you are to construct and use mathematical models. Remember that the most important thing is getting the particular equation correct. With an incorrect equation, other parts of the problem may turn out to be senseless. Be alert to the fact that some of the problems call for a *linear* function, not a quadratic function!

EXERCISE 5-7

Do These Quickly

The following problems are intended to refresh your skills. You should be able to do all 10 in less than 5 minutes.

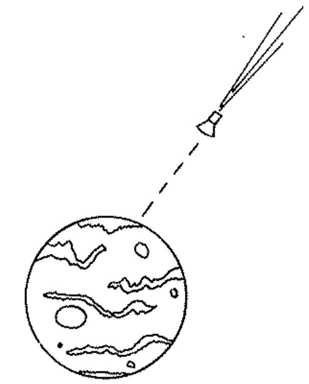
Q1. Write the general equation for a linear function.

Q2. Write the general equation for a quadratic function.

Q3. Does the graph of $y = 15 + 7x - 5x^2$ open upward or downward?

Q4. Does the graph of $y = 3x - 5$ have positive slope or negative slope?

Q5. Find the discriminant: $x^2 + 3x + 12 = 2$ (Beware!)



Q6. Solve for a: $3a + b = 17$
$5a + b = 10$

Q7. Do the squaring: $(x + 7)^2$

Q8. Factor: $x^2 + 12x + 20$

Q9. Draw a rhombus.

Q10. Multiply: $\left(\frac{3}{11}\right)\left(\frac{22}{7}\right)$

Work the following problems.

1. ***Phoebe Small's Rocket Problem*** Phoebe Small is out Sunday driving in her spaceship. As she approaches Mars, she changes her mind, decides that she does not wish to visit that planet, and fires her retro-rocket. The spaceship slows down, and if all goes well, stops for an instant then starts pulling away. While the rocket motor is firing, Phoebe's distance, d, from the surface of Mars depends by a quadratic function on the number of minutes, t, since she started firing the rocket.
   a. Phoebe finds that at times $t = 1$, 2, and 3 minutes, her distances are $d = 425$, 356, and 293 kilometers, respectively. Find the particular equation expressing d in terms of t.
   b. Find the d-intercept and tell what this number represents in the real world.
   c. According to the equation, where will Phoebe be when $t = 15$? When $t = 16$? Does this tell you she is pulling away from Mars when $t = 16$, or still approaching?
   d. Does your model tell you that Phoebe crashed into the surface of Mars, just touches the surface, or pulls away before reaching the surface? Explain.
   e. Draw the graph of this quadratic function. Show the vertex.
   f. Based on your answers to the above questions, in what domain do you think this quadratic function will give reasonable values for d? Modify your graph in part e, if necessary, by using a *dotted* line for those parts of the graph that are out of this domain.

2. ***Bathtub Problem*** Assume that the number of liters of water remaining in the bathtub varies quadratically with the number of minutes which have elapsed since you pulled the plug.
   a. If the tub has 38.4, 21.6, and 9.6 liters remaining at 1, 2, and 3 minutes respectively, since you pulled the plug, write an equation expressing liters in terms of time.
   b. How much water was in the tub when you pulled the plug?
   c. When will the tub be empty?
   d. In the real world, the number of liters would never be negative. What is the lowest number of liters the *model* predicts? Is this number reasonable?